Theoretical 2: Solution
Relativistic Correction on GPS Satelitte

Part A. Single accelerated particle

1. The equation of motion is given by

$$F = \frac{d}{dt}(\gamma m v) \tag{1}$$
$$= \frac{mc\dot{\beta}}{(1-\beta^2)^{\frac{3}{2}}}$$
$$F = \gamma^3 m a, \tag{2}$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and $\beta = \frac{v}{c}$. So the acceleration is given by

$$a = \frac{F}{\gamma^3 m}. \tag{3}$$

2. Consider the following systems, a frame S' is moving with respect to another frame S, with velocity u in the x direction. If a particle is moving in the S' frame with velocity v also in x direction, then the particle velocity in the S frame is given by

$$v = \frac{u+v'}{1+\frac{uv'}{c^2}}. \tag{4}$$

If the particles velocity changes with respect to the S' frame, then the velocity in the S frame is also change according to

$$dv = \frac{dv'}{1+\frac{uv'}{c^2}} - \frac{u+v'}{\left(1+\frac{uv'}{c^2}\right)^2}\frac{udv'}{c^2}$$
$$dv = \frac{1}{\gamma^2}\frac{dv'}{\left(1+\frac{uv'}{c^2}\right)^2}. \tag{5}$$

The time in the S' frame is t', so the time in the S frame is given by

$$t = \gamma\left(t' + \frac{ux'}{c^2}\right), \tag{6}$$

so the time change in the S' frame will give a time change in the S frame as follow

$$dt = \gamma dt'\left(1+\frac{uv'}{c^2}\right). \tag{7}$$

The acceleration in the S frame is given by

$$a = \frac{dv}{dt} = \frac{a'}{\gamma^3}\frac{1}{\left(1+\frac{uv'}{c^2}\right)^3}. \tag{8}$$

If the S' frame is the proper frame, then by definition the velocity $v = 0$. Substitute this to the last equation, we get

$$a = \frac{a'}{\gamma^3}. \tag{9}$$

Combining Eq.(3) and Eq.(9), we get

$$a' = \frac{F}{m} \equiv g. \tag{10}$$

3. Eq.(9) can be rewrite as

$$c\frac{d\beta}{dt} = \frac{g}{\gamma^3}$$

$$\int_0^\beta \frac{d\beta}{(1-\beta^2)^{\frac{3}{2}}} = \frac{g}{c}\int_0^t dt$$

$$\frac{\beta}{\sqrt{1-\beta^2}} = \frac{gt}{c} \tag{11}$$

$$\beta = \frac{\frac{gt}{c}}{\sqrt{1+\left(\frac{gt}{c}\right)^2}}. \tag{12}$$

4. Using Eq.(12), we get

$$\int_0^x dx = \int_0^t \frac{gtdt}{\sqrt{1+\left(\frac{gt}{c}\right)^2}}$$

$$x = \frac{c^2}{g}\left(\sqrt{1+\left(\frac{gt}{c}\right)^2} - 1\right). \tag{13}$$

5. Eq.(3) can also be rewrite as follow

$$c\frac{d\beta}{\gamma d\tau} = \frac{g}{\gamma^3} \tag{14}$$

$$\int_0^\beta \frac{d\beta}{1-\beta^2} = \frac{g}{c}\int_0^t d\tau$$

$$\ln\left(\frac{1}{\sqrt{1-\beta^2}} + \frac{\beta}{\sqrt{1-\beta^2}}\right) = \frac{g\tau}{c} \tag{15}$$

$$\sqrt{\frac{1+\beta}{1-\beta}} = e^{\frac{g\tau}{c}}$$

$$\beta\left(e^{\frac{g\tau}{c}} + e^{-\frac{g\tau}{c}}\right) = e^{\frac{g\tau}{c}} - e^{-\frac{g\tau}{c}}$$

$$\beta = \tanh\frac{g\tau}{c}. \tag{16}$$

6. The time dilation relation is
$$dt = \gamma d\tau. \tag{17}$$
From eq.(16), we have
$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \cosh\frac{g\tau}{c}. \tag{18}$$
Combining this equations, we get
$$\int_0^t dt = \int_0^\tau d\tau \cosh\frac{g\tau}{c}$$
$$t = \frac{c}{g}\sinh\frac{g\tau}{c}. \tag{19}$$

Part B. Minkowski Diagram

1. The following figure shows the situation

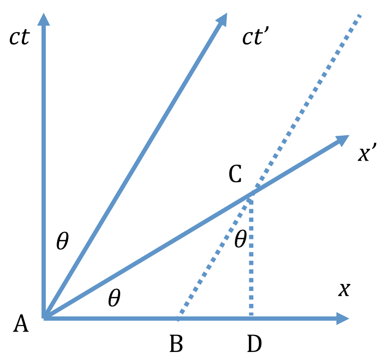

Figure 1: Minkowski Diagram

where $\tan\theta = \beta$, $\sin\theta = \frac{\beta}{\sqrt{1+\beta^2}}$, and $\cos\theta = \frac{1}{\sqrt{1+\beta^2}}$.
The line AC represents the stick with proper length equal L in the S' frame.
The length AC is equal to $\sqrt{\frac{1+\beta^2}{1-\beta^2}}L$ in the S frame.
The stick length in the S frame is represented by the line AB

$$\begin{aligned}\mathrm{AB} &= \mathrm{AD} - \mathrm{BD}\\ &= \mathrm{AC}\cos\theta - \mathrm{AC}\sin\theta\tan\theta\\ &= \sqrt{\frac{1+\beta^2}{1-\beta^2}}L\frac{1}{\sqrt{1+\beta^2}} - \sqrt{\frac{1+\beta^2}{1-\beta^2}}L\frac{\beta}{\sqrt{1+\beta^2}}\beta\\ \mathrm{AB} &= \sqrt{1-\beta^2}L. \end{aligned} \tag{20}$$

This result is the same with the result from Lorentz transformation calculation.

2. The figure below show the setting of the problem.

   The line AB represents the stick with proper length equal L in the S frame.
   The length AB is equal to $\sqrt{\frac{1-\beta^2}{1+\beta^2}}L$ in the S' frame.
   The stick length in the S' frame is represented by the line AC

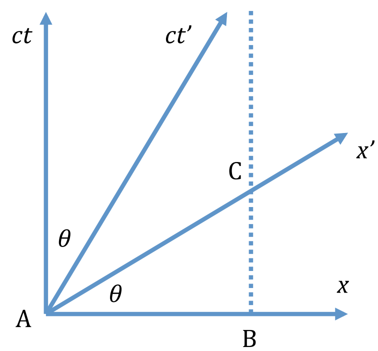

Figure 2: Minkowski Diagram

$$\text{AC} = \frac{\text{AB}}{\cos\theta} = \sqrt{1-\beta^2}L. \tag{21}$$

3. The position of the particle is given by eq.(13).

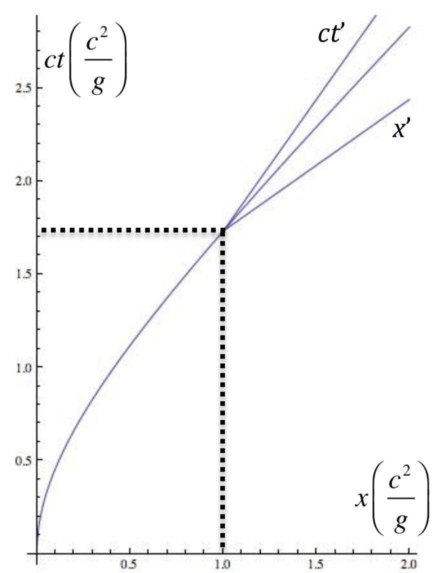

Figure 3: Minkowski Diagram

Part C. Flight Time

1. When the clock in the origin time is equal to t_0, it emits a signal that contain the information of its time. This signal will arrive at the particle at time t, while the particle position is at $x(t)$. We have

$$c(t-t_0) = x(t) \tag{22}$$
$$t - t_0 = \frac{c}{g}\left(\sqrt{1+\left(\frac{gt}{c}\right)^2} - 1\right)$$
$$t = \frac{t_0}{2}\frac{2-\frac{gt_0}{c}}{1-\frac{gt_0}{c}}. \tag{23}$$

When the information arrive at the particle, the particle's clock has a reading according to eq.(19). So we get

$$\frac{c}{g}\sinh\frac{g\tau}{c} = \frac{t_0}{2}\frac{2-\frac{gt_0}{c}}{1-\frac{gt_0}{c}}$$
$$0 = \frac{1}{2}\left(\frac{gt_0}{c}\right)^2 - \frac{gt_0}{c}\left(1+\sinh\frac{g\tau}{c}\right) + \sinh\frac{g\tau}{c}$$
$$\frac{gt_0}{c} = 1 + \sinh\frac{g\tau}{c} \pm \cosh\frac{g\tau}{c}. \tag{24}$$

Using initial condition $t = 0$ when $\tau = 0$, we choose the negative sign

$$\frac{gt_0}{c} = 1 + \sinh\frac{g\tau}{c} - \cosh\frac{g\tau}{c}$$
$$t_0 = \frac{c}{g}\left(1 - e^{-\frac{g\tau}{c}}\right). \tag{25}$$

As $\tau \to \infty$, $t_0 = \frac{c}{g}$. So the clock reading will freeze at this value.

2. When the particles clock has a reading τ_0, its position is given by eq.(13), and the time t_0 is given by eq.(19). Combining this two equation, we get

$$x = \frac{c^2}{g}\left(\sqrt{1+\sinh^2\frac{g\tau_0}{c}} - 1\right). \tag{26}$$

The particle's clock reading is then sent to the observer at the origin. The total time needed for the information to arrive is given by

$$t = \frac{c}{g}\sinh\frac{g\tau_0}{c} + \frac{x}{c} \tag{27}$$
$$= \frac{c}{g}\left(\sinh\frac{g\tau_0}{c} + \cosh\frac{g\tau_0}{c} - 1\right)$$
$$t = \frac{c}{g}\left(e^{\frac{g\tau_0}{c}} - 1\right) \tag{28}$$
$$\tau_0 = \frac{c}{g}\ln\left(\frac{gt}{c} + 1\right). \tag{29}$$

The time will not freeze.

Part D. Two Accelerated Particles

1. $\tau_2 = \tau_1$.

2. From the diagram, we have

$$\tan\theta = \beta = \frac{ct_2 - ct_1}{x_2 - x_1}. \tag{30}$$

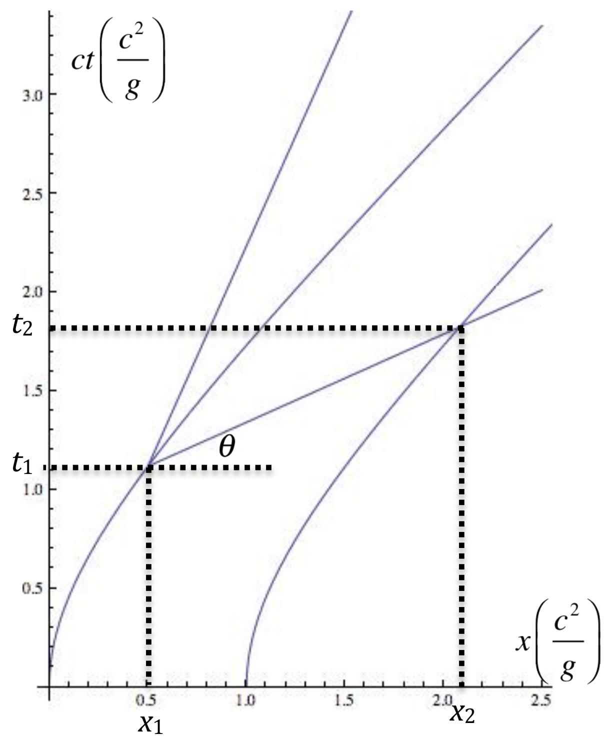

Figure 4: Minkowski Diagram for two particles

Using eq.(13), and eq.(19) along with the initial condition, we get

$$x_1 = \frac{c^2}{g}\left(\cosh\frac{g\tau_1}{c} - 1\right), \tag{31}$$

$$x_2 = \frac{c^2}{g}\left(\cosh\frac{g\tau_2}{c} - 1\right) + L. \tag{32}$$

Using eq.(16), eq.(19), eq.(31) and eq.(32), we obtain

$$\begin{aligned} \tanh \frac{g\tau_1}{c} &= \frac{c\left(\frac{c}{g}\sinh\frac{g\tau_2}{c} - \frac{c}{g}\sinh\frac{g\tau_1}{c}\right)}{L + \frac{c^2}{g}\left(\cosh\frac{g\tau_2}{c} - 1\right) - \frac{c^2}{g}\left(\cosh\frac{g\tau_1}{c} - 1\right)} \\ &= \frac{\sinh\frac{g\tau_2}{c} - \sinh\frac{g\tau_1}{c}}{\frac{gL}{c^2} + \cosh\frac{g\tau_2}{c} - \cosh\frac{g\tau_1}{c}} \\ \frac{gL}{c^2}\sinh\frac{g\tau_1}{c} &= \sinh\frac{g\tau_2}{c}\cosh\frac{g\tau_1}{c} - \cosh\frac{g\tau_2}{c}\sinh\frac{g\tau_1}{c} \\ \frac{gL}{c^2}\sinh\frac{g\tau_1}{c} &= \sinh\frac{g}{c}\left(\tau_2 - \tau_1\right). \end{aligned} \tag{33}$$

So $C_1 = \frac{gL}{c^2}$.

3. From the length contraction, we have

$$L' = \frac{x_2 - x_1}{\gamma_1} \tag{34}$$

$$\frac{dL'}{d\tau_1} = \left(\frac{dx_2}{d\tau_2}\frac{d\tau_2}{d\tau_1} - \frac{dx_1}{d\tau_1}\right)\frac{1}{\gamma_1} - \frac{x_2 - x_1}{\gamma_1^2}\frac{d\gamma_1}{d\tau_1}. \tag{35}$$

Take derivative of eq.(31), eq.(32) and eq.(33), we get

$$\frac{dx_1}{d\tau_1} = c\sinh\frac{g\tau_1}{c}, \tag{36}$$

$$\frac{dx_2}{d\tau_2} = c\sinh\frac{g\tau_2}{c}, \tag{37}$$

$$\frac{gL}{c^2}\cosh\frac{g\tau_1}{c} = \cosh\frac{g}{c}\left(\tau_2 - \tau_1\right)\left(\frac{d\tau_2}{d\tau_1} - 1\right). \tag{38}$$

The last equation can be rearrange to get

$$\frac{d\tau_2}{d\tau_1} = \frac{\frac{gL}{c^2}\cosh\frac{g\tau_1}{c}}{\cosh\frac{g}{c}\left(\tau_2 - \tau_1\right)} + 1. \tag{39}$$

From eq.(30), we have

$$x_2 - x_1 = \frac{c(t_2 - t_1)}{\beta_1} = \frac{c}{\tanh\frac{g\tau_1}{c}}\left(\frac{c}{g}\sinh\frac{g\tau_2}{c} - \frac{c}{g}\sinh\frac{g\tau_1}{c}\right). \tag{40}$$

Combining all these equations, we get

$$\begin{aligned} \frac{dL_1}{d\tau_1} &= \left(c\sinh\frac{g\tau_2}{c}\frac{\frac{gL}{c^2}\cosh\frac{g\tau_1}{c}}{\cosh\frac{g}{c}\left(\tau_2 - \tau_1\right)} + c\sinh\frac{g\tau_2}{c} - c\sinh\frac{g\tau_1}{c}\right)\frac{1}{\cosh\frac{g\tau_1}{c}} \\ &\quad - \frac{c^2}{g}\left(\sinh\frac{g\tau_2}{c} - \sinh\frac{g\tau_1}{c}\right)\frac{1}{\tanh\frac{g\tau_1}{c}}\frac{1}{\cosh^2\frac{g\tau_1}{c}}\frac{g}{c}\sinh\frac{g\tau_1}{c} \\ \frac{dL_1}{d\tau_1} &= \frac{gL}{c}\frac{\sinh\frac{g\tau_2}{c}}{\cosh\frac{g}{c}\left(\tau_2 - \tau_1\right)}. \end{aligned} \tag{41}$$

So $C_2 = \frac{gL}{c}$.

Part E. Uniformly Accelerated Frame

1. Distance from a certain point x_p according to the particle's frame is

$$L' = \frac{x - x_p}{\gamma} \tag{42}$$

$$L' = \frac{\frac{c^2}{g_1}\left(\cosh\frac{g_1\tau}{c} - 1\right) - x_p}{\cosh\frac{g_1\tau}{c}}$$

$$L' = \frac{c^2}{g_1} - \frac{\frac{c^2}{g_1} + x_p}{\cosh\frac{g_1\tau}{c}}. \tag{43}$$

For L' equal constant, we need $x_p = -\frac{c^2}{g_1}$.

2. **First method:** If the distance in the S' frame is constant $= L$, then in the S frame the length is

$$L_s = L\sqrt{\frac{1+\beta^2}{1-\beta^2}}. \tag{44}$$

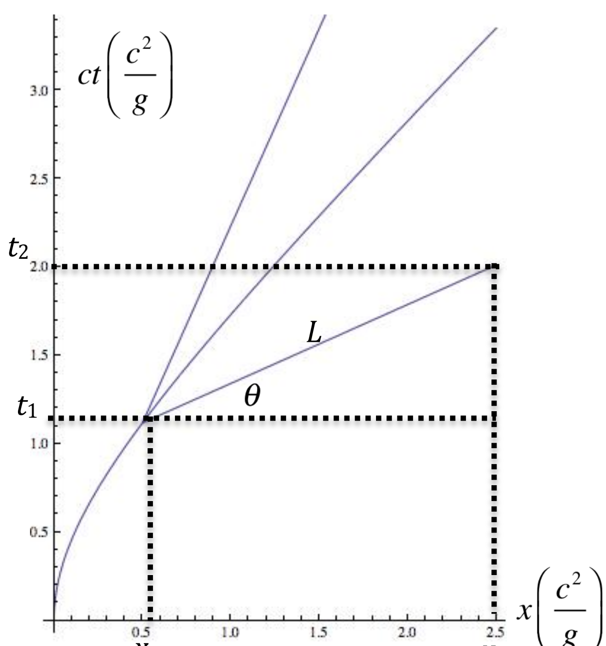

Figure 5: Minkowski Diagram for two particles

So the position of the second particle is

$$x_2 = x_1 + L_s \cos\theta \tag{45}$$
$$= \frac{c^2}{g_1}\left(\sqrt{1+\left(\frac{g_1 t_1}{c}\right)^2} - 1\right) + L\sqrt{1+\left(\frac{g_1 t_1}{c}\right)}$$
$$x_2 = \left(\frac{c^2}{g_1} + L\right)\sqrt{1+\left(\frac{g_1 t_1}{c}\right)^2} - \frac{c^2}{g_1}. \tag{46}$$

The time of the second particle is

$$ct_2 = ct_1 + L_s \sin\theta \tag{47}$$
$$= ct_1 + L\sqrt{\frac{1+\beta^2}{1-\beta^2}}\frac{\beta}{\sqrt{1+\beta^2}}$$
$$ct_2 = t_1\left(c + \frac{g_1 L}{c}\right). \tag{48}$$

Substitute eq.(48) to eq.(46) to get

$$x_2 = \left(\frac{c^2}{g_1} + L\right)\sqrt{1+\left(\frac{g_1}{c}\frac{t_2}{1+\frac{g_1 L}{c^2}}\right)^2} - \frac{c^2}{g_1}$$
$$x_2 = \left(\frac{c^2}{g_1} + L\right)\sqrt{1+\left(\frac{g_1}{1+\frac{g_1 L}{c^2}}\frac{t_2}{c}\right)^2} - \frac{c^2}{g_1}. \tag{49}$$

From the last equation, we can identify

$$g_2 \equiv \frac{g_1}{1+\frac{g_1 L}{c^2}}. \tag{50}$$

As for confirmation, we can subsitute this relation to the second particle position to get

$$x_2 = \frac{c^2}{g_2}\sqrt{1+\left(\frac{g_2 t_2}{c}\right)^2} - \frac{c^2}{g_1}. \tag{51}$$

Second method: In this method, we will choose g_2 such that the special point like the one descirbe in the question 1 is exactly the same as the similar point for the proper acceleration g_1.
For first particle, we have $x_{p1} g_1 = c^2$
For second particle, we have $(L + x_{p1}) g_2 = c^2$

Combining this two equations, we get

$$g_2 = \frac{c^2}{L + \frac{c^2}{g_1}}$$
$$g_2 = \frac{g_1}{1 + \frac{g_1 L}{c^2}}. \tag{52}$$

3. The relation between the time in the two particles is given by eq.(48)

$$t_2 = t_1 \left(1 + \frac{g_1 L}{c^2}\right)$$
$$\frac{c^2}{g_2} \sinh \frac{g_2 \tau_2}{c} = \frac{c^2}{g_1} \sinh \frac{g_1 \tau_1}{c} \left(1 + \frac{g_1 L}{c^2}\right)$$
$$\sinh \frac{g_2 \tau_2}{c} = \sinh \frac{g_1 \tau_1}{c}$$
$$g_2 \tau_2 = g_1 \tau_1 \tag{53}$$
$$\frac{d\tau_2}{d\tau_1} = \frac{g_1}{g_2} = 1 + \frac{g_1 L}{c^2}. \tag{54}$$

Part F. Correction for GPS

1. From Newtons Law

$$\frac{GMm}{r^2} = m\omega^2 r \tag{55}$$
$$r = \left(\frac{gR^2 T^2}{4\pi^2}\right)^{\frac{1}{3}} \tag{56}$$
$$r = 2.66 \times 10^7 \text{m}.$$

The velocity is given by

$$v = \omega r = \left(\frac{2\pi g R^2}{T}\right)^{\frac{1}{3}} \tag{57}$$
$$= 3.87 \times 10^3 \text{m/s}.$$

2. The general relativity effect is

$$\frac{d\tau_g}{dt} = 1 + \frac{\Delta U}{mc^2} \tag{58}$$
$$\frac{d\tau_g}{dt} = 1 + \frac{gR^2}{c^2} \frac{R - r}{Rr}. \tag{59}$$

After one day, the difference is

$$\begin{aligned}\Delta\tau_g &= \frac{gR^2}{c^2}\frac{R-r}{Rr}\Delta T \\ &= 4.55\times 10^{-5}\text{s.}\end{aligned} \tag{60}$$

The special relativity effect is

$$\frac{d\tau_s}{dt} = \sqrt{1-\frac{v^2}{c^2}} \tag{61}$$

$$= \sqrt{1-\left(\left(\frac{2\pi gR^2}{T}\right)^{\frac{2}{3}}\right)\frac{1}{c^2}}$$

$$\approx 1-\frac{1}{2}\left(\left(\frac{2\pi gR^2}{T}\right)^{\frac{2}{3}}\right)\frac{1}{c^2}. \tag{62}$$

After one day, the difference is

$$\begin{aligned}\Delta\tau_s &= -\frac{1}{2}\left(\left(\frac{2\pi gR^2}{T}\right)^{\frac{2}{3}}\right)\frac{1}{c^2}\Delta T \\ &= -7.18\times 10^{-6}\text{s.}\end{aligned} \tag{63}$$

The satelite's clock is faster with total $\Delta\tau = \Delta\tau_g + \Delta\tau_s = 3.83\times 10^{-5}$s.

3. $\Delta L = c\Delta\tau = 1.15\times 10^4\text{m} = 11.5\text{km}$.