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Solutions to Theory Problem 3

Physics of Live Systems

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v1.0

Physics of Live Systems (10 points)

Part A. The physics of blood flow (4.5 points)

A.1

Since the vessel network is symmetrical, the flow in a vessel of level $i+1$ is half the flow in a vessel of level i.

In this way, we can sum the pressure differences in all levels:

$$\Delta P = \sum_{i=0}^{N-1} Q_i R_i = Q_0 \sum_{i=0}^{N-1} \frac{R_i}{2^i}.$$

Introducing the radii dependences yields

$$\Delta P = Q_0 \sum_{i=0}^{N-1} \frac{8\ell_i \eta}{2^i \pi r_i^4} = Q_0 \frac{8\ell_0 \eta}{\pi r_0^4} \sum_{i=0}^{N-1} \frac{2^{4i/3}}{2^i 2^{i/3}} = Q_0 N \frac{8\ell_0 \eta}{\pi r_0^4}\ .$$

Therefore

$$Q_0 = \Delta P \frac{\pi r_0^4}{8N\ell_0 \eta}\ .$$

Hence, the flow rate for a vessel network in level i is

A.1	$$Q_i = \Delta P \frac{\pi r_0^4}{2^{i+3} N \ell_0 \eta}\ .$$	1.3pt

A.2

Replace values in the formula and change units appropriately

$$\begin{aligned} Q_0 &= \frac{\Delta P \pi r_0^4}{8N\ell_0\eta} = \\ &= \frac{(55-30)\times 1.013\times 10^5 \times 3.1415 \times (6.0\times 10^{-5})^4}{760 \times 48 \times 2.0 \times 10^{-3} \times 3.5 \times 10^{-3}} = 4.0\times 10^{-10}\ \text{m}^3/\text{s} \end{aligned}$$

to obtain the final value in the requested unites:

A.2	$$Q_0 \simeq 1.5\ \text{m}\ell/\text{h}\ .$$	0.5pt

A.3

The current is given by

$$I = \frac{P_{\text{in}} e^{i\omega t}}{R + i\omega L + \frac{1}{i\omega C}} \,.$$

The pressure difference in the capacitor is

$$P_{\text{out}} e^{i(\omega t + \phi)} = \frac{P_{\text{in}} e^{i\omega t}}{R + i\omega L + \frac{1}{i\omega C}} \, \frac{1}{i\omega C} = \frac{P_{\text{in}} e^{i\omega t}}{i\omega C R - \omega^2 L C + 1} \,.$$

The amplitude is

$$P_{\text{out}} = \frac{P_{\text{in}}}{\sqrt{(1-\omega^2 LC)^2 + \omega^2 C^2 R^2}} \,.$$

To be smaller than P_{in}, for $\omega \to 0$:

$$(1-\omega^2 LC)^2 + \omega^2 C^2 R^2 > 1 \iff -2CL + C^2 R^2 > 0 \,.$$

Replacing the expressions for L, C, and R we get: $\frac{64\eta^2 \ell^2}{3Ehr^3\rho} > 1$.

A.3 2.0pt

$$P_{\text{out}} = \frac{P_{\text{in}}}{\sqrt{(1-\omega^2 LC)^2 + \omega^2 C^2 R^2}} \,.$$

Condition:

$$\frac{64\eta^2 \ell^2}{3Ehr^3\rho} > 1 \,.$$

Alternative way to obtain P_{out}:

The amplitude of the current in the equivalent circuit is $I_0 = \frac{P_{\text{in}}}{Z}$, where

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

is the modulus of the impedance. Hence, the voltage amplitude in the capacitor is

$$P_{\text{out}} = \frac{1}{\omega C} \times I_0 = \frac{P_{\text{in}}}{\sqrt{\omega^2 C^2 R^2 + (\omega^2 LC - 1)^2}}.$$

A.4

The previous condition can also be expressed as

$$h < \frac{64\eta^2 \ell^2}{3Er^3\rho} \,.$$

For the network referred to in **A.2**

$$h < \frac{64\eta^2 \ell_0^2 \times 2^i}{3 \times 2^{2i/3} E r_0^3 \rho} = \frac{64 \times (3.5\times10^{-3})^2 \times (2.0\times10^{-3})^2}{3 \times 0.06 \times 10^6 \times (6.0\times10^{-5})^3 \times 1.05\times10^3} \times 2^{i/3} = 7.7\times10^{-5} \times 2^{i/3} \,.$$

For $i = 0$, in the worse case scenario,

$$h_{\text{max}} = 7.7 \times 10^{-5} \times 2^0 = 7.7 \times 10^{-5} \text{ m}$$

This value is certainly observed in these vessels since their radius range from 18 μm to 60 μm. A wall width smaller than 80 μm is certainly reasonable.

A.4	Maximum $h = 8 \times 10^{-5}$ m	0.7pt

Part B. Tumor growth (5.5 points)

B.1

The expressions for the masses of tumour and normal tissue are written as:

$$\begin{cases} M_{\text{T}} = V_{\text{T}}\rho_{\text{T}} = V_{\text{T}}\rho_0(1 + \frac{p}{K_{\text{T}}}) \\ \\ M_{\text{N}} = V\rho_0 = (V - V_{\text{T}})\rho_0(1 + \frac{p}{K_{\text{N}}}) \end{cases}$$

The pressure, p, can be expressed as

$$p = \frac{M_{\text{T}}\, K_{\text{T}}}{V_{\text{T}}\, \rho_0} - K_{\text{T}}$$

and, then, used in the equation for M_{N}:

$$M_{\text{N}} = (V - V_{\text{T}})\, \frac{M_{\text{N}}}{V} \left[\left(1 - \frac{K_{\text{T}}}{K_{\text{N}}} \right) + \frac{M_{\text{T}}\, V K_{\text{T}}}{V_{\text{T}}\, M_{\text{N}}\, K_{\text{N}}} \right]$$

Simplifying and rearranging the terms, the equation for v becomes

$$(1 - \kappa)\; v^2 - (1 + \mu)\; v + \mu = 0\,,$$

for which the solution is (the other solution of the quadratic equation is not physically relevant since does not lead to $v = 0$ for $\mu = 0$)

B.1	$$v = \frac{1 + \mu - \sqrt{(1+\mu)^2 - 4\mu\,(1-\kappa)}}{2(1-\kappa)}\,.$$	1.0pt

B.2

For $r < R_{\text{T}}$, the conservation of energy implies that

$$4\pi r^2 (-k) \frac{\mathrm{d}T}{\mathrm{d}r} = \mathcal{P} \frac{4}{3}\pi r^3\,.$$

Therefore, the temperature difference to 37 °C = 310.15 K, $\Delta T(r)$, is given by

$$\Delta T(r) = -\frac{\mathcal{P} r^2}{6k} + C ,$$

where C is a constant.

For $r > R_\mathsf{T}$, the conservation of energy implies that

$$4\pi r^2 (-k) \frac{\mathrm{d}T}{\mathrm{d}r} = \mathcal{P} \frac{4}{3} \pi R_\mathsf{T}^3 .$$

Therefore, the temperature difference to 37 °C is

$$\Delta T(r) = \frac{\mathcal{P} R_\mathsf{T}^3}{3kr} .$$

In this case there is no constant, since very far away the increase in temperature is zero.

Matching the two solutions at $r = R_\mathsf{T}$ gives

$$C = \frac{\mathcal{P} R_\mathsf{T}^2}{2k} .$$

Therefore the temperature at the centre of the tumour, in SI units, is

B.2	Temperature: $310.15 + \frac{\mathcal{P} R_\mathsf{T}^2}{2k}$.	1.7pt

B.3

The increase in temperature at the tumour surface (the lower temperature in the tumour) is

$$\Delta T(R_\mathsf{T}) = \frac{\mathcal{P} R_\mathsf{T}^2}{3k} .$$

This increase should be equal to 6.0 K. Therefore,

$$\mathcal{P} = \frac{3 \Delta T k}{R_\mathsf{T}^2} = \frac{3 \times 6 \times 0.6}{0.05^2} = 4.3 \text{ kW/m}^3 .$$

B.3	$\mathcal{P}_\mathsf{min} = 4.3$ kW/m^3.	0.5pt

B.4

We can relate δr with the pressure in the tumour, using the relation given in the text up to leading order in $p - P_\mathsf{cap}$: $\delta r = \frac{p - P_\mathsf{cap}}{2(p_\mathsf{c} - P_\mathsf{cap})} \delta r_\mathsf{c}$. Therefore, if $p - P_\mathsf{cap}$ is very small, also it is δr.

The pressure can be related with the volume. We know that

$$\frac{M_\mathsf{N}}{V_\mathsf{N}} = \frac{\rho_0 V}{V - V_\mathsf{T}} = \frac{\rho_0}{1 - v} = \rho_0 \left(1 + \frac{p}{K_\mathsf{N}} \right) .$$

And so $p = \frac{K_\mathsf{N} v}{1-v}$.

When the thinner vessels are narrower, the flow rate in the main vessel is altered:

$$\Delta P = (Q_0 + \delta Q_0) \sum_{i=0}^{N-1} \frac{8\ell_i \eta}{2^i \pi r_i^4} = (Q_0 + \delta Q_0) \frac{8\ell_0 \eta}{\pi r_0^4} \left(\sum_{i=0}^{N-2} \frac{2^{4i/3}}{2^i 2^{i/3}} + \frac{2^{4(N-1)/3}}{2^{N-1} 2^{(N-1)/3} \left(1 - \frac{\delta r}{r_0/2^{(N-1)/3}}\right)^4} \right)$$

$$\implies \Delta P \simeq (Q_0 + \delta Q_0) \frac{\Delta P}{N Q_0} \left(N - 1 + 1 + \frac{4\,\delta r}{r_{N-1}} \right)$$

Noting that $\frac{\delta Q_{N-1}}{Q_{N-1}} = \frac{\delta Q_0}{Q_0}$, we obtain

$$1 + \frac{\delta Q_{N-1}}{Q_{N-1}} = \frac{1}{1 + \frac{4\,\delta r}{N r_{N-1}}} \simeq 1 - \frac{4\,\delta r}{N r_{N-1}}\,.$$

And so:

$$\frac{\delta Q_{N-1}}{Q_{N-1}} \simeq -\frac{4}{N} \frac{\delta r}{r_{N-1}}\,.$$

Putting all together

B.4 2.3pt

$$\frac{\delta Q_{N-1}}{Q_{N-1}} \simeq -\frac{2}{N} \frac{K_\mathsf{N} v - (1-v) P_\mathsf{cap}}{(1-v)(p_\mathsf{c} - P_\mathsf{cap})} \frac{\delta r_\mathsf{c}}{r_{N-1}}\,.$$