Vortices in Superfluid

MODB

2017-05-03

Introduction

Superfluidity is a property of flowing without friction. Everyday experience tells us that motion of an ordinary fluid (say, water at room temperature) is always accompanied by viscous dissipation of energy, so that the flow gradually becomes slower unless it is maintained by external forces. In contrast, superfluid exhibits no loss of kinetic energy: once excited the motion of superfluid can continue indefinitely. Superfluidity was originally discovered experimentally in liquid helium.

We study properties of superfluid helium at zero temperature. It will be treated as an incompressible fluid with density ρ. Flow continuity (the fact that the mass flowing into and the mass flowing out of a given infinitesimal volume are equal) implies that the flux of helium velocity $\vec{v}$ through a closed surface is always zero. Superfluid velocity in this aspect is similar to the magnetic field intensity. Similarly to the magnetic field lines, "streamlines" are tangential to the fluid velocity at each point and their density is proportional to its magnitude.

True superflow has an important property of being irrotational: circulation of superfluid velocity $\vec{v}$ along any closed path within helium is zero

$$\int_L \vec{v} \cdot d\vec{l} = 0.$$

This statement must be amended if superfluidity is absent along a thin "vortex filament". The thickness of the filament itself is of approximately atomic dimensions a, but the vortex induces long range velocity field in surrounding superfluid: velocity circulation around such filament is the circulation quantum[1]

$$\left| \int_L \vec{v} \cdot d\vec{l} \right| = 2\pi\kappa, \tag{1}$$

[1]Circulation quantization is a macroscopic quantum effect and corresponds to the angular momentum quantization in Bohr model. The circulation quantum can be expressed as $\kappa = \hbar/m_{\mathrm{He}}$, where m_{He} is the mass of helium atom.

and zero if the path can be contracted to a single point without crossing the vortex, see Fig. 1. This supports the analogy between superflow and magnetic field: superposition of two valid velocity distributions is a valid velocity distribution and the velocity at any point is equal (up to a dimensional factor) to the magnetic field produced by the unit currents running through a system of wires representing vortex filaments.

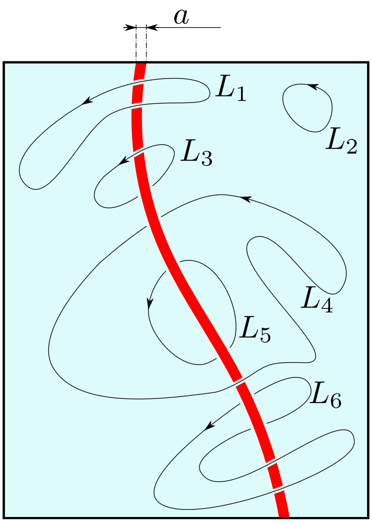


Figure 1: Vortex filament (red) in superfluid (light blue). Velocity circulations along paths L_1, L_2, L_5, and L_6 are all zero, but those for L_3 and L_4 are equal to $\pm 2\pi\kappa$. Note that circulations along L_3 and L_4 have opposite signs.

A. Steady filament (0.75)

Consider a cylindrical beaker (radius $R_0 \gg a$) of superfluid helium and a straight vertical vortex filament in its center Fig. 2.

A1 (0.25)

Plot the streamlines. Find out the velocity v at a point $\vec{r}$.

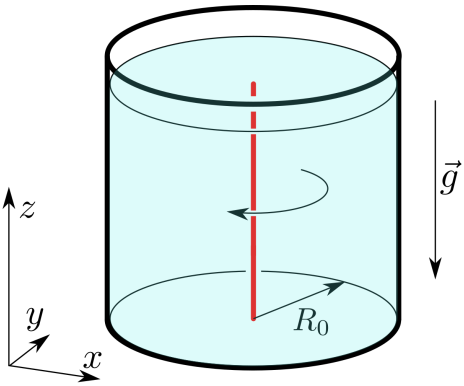
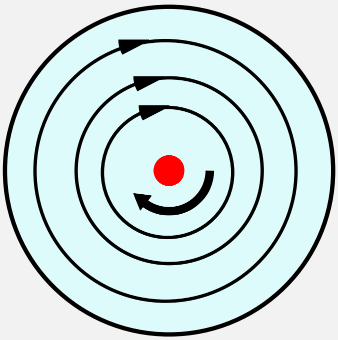


Figure 2: Straight vortex along the axis of a beaker.



The streamlines are circular. From the circulation identity (1) it is obvious that $v = \kappa/r$.

- Streamlines are plotted correctly 0.1
- $v = \frac{k}{r}$ 0.15

A2 (0.5)

Work out the free surface shape (height as a function of coordinate $z(\vec{r})$) around the vortex. Free fall acceleration is g, surface tension can be neglected.

Consider a thin circular layer of the radius r. Equilibrium condition for its surface is given by the requirement

$$g\frac{dz}{dr} = \frac{v^2}{r} = \frac{\kappa^2}{r^3}. \tag{2}$$

This equation is satisfied by the surface profile

$$z(r) = -\frac{\kappa^2}{2gr^2}. \tag{3}$$

- $\tan\alpha = \frac{k^2}{gr^3}$ or equivalent 0.25
- $z = -\frac{k^2}{2gr^2}$ 0.25

B. Vortex motion (1.25)

Free vortices move about in space with the flow[2]. In other words each element of the filament moves with the velocity $\vec{v}$ of the fluid at the position of that element.

As an example, consider a pair of counter-rotating straight vortices placed initially at distance r_0 from each other, see Fig. 3. Each vortex produces velocity $v_0 = \kappa/r_0$ at the axis of another. As a result, the vortex pair moves rectilinearly with constant speed $v_0 = \kappa/r_0$ so that the distance between them remains unchanged.

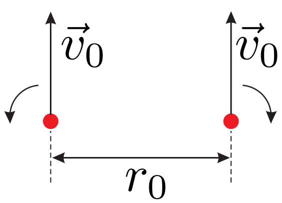


Figure 3: Parallel vortex filaments with opposite circulations

B1 (0.25)

Consider two identical straight vortices initially placed at distance r_0 from each other as shown in Fig. 4. Find initial velocities of the vortices and draw their trajectories.

[2]This is a consequence of momentum conservation, see next section.

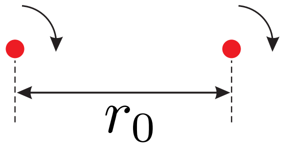
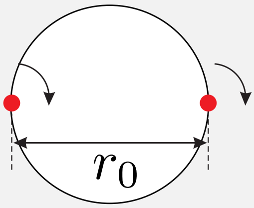


Figure 4: Parallel vortex filaments with equal circulations



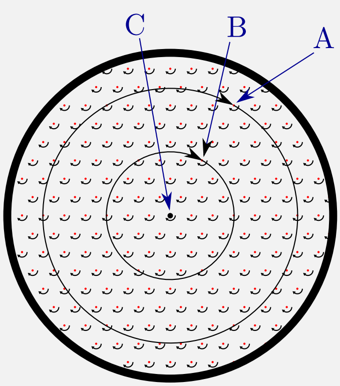
Being advected by each other's flow field, filaments will rotate around a point halfway between them. The velocity is given by $v_0 = \kappa/r_0$.

- Trajectories are plotted correctly 0.15
- Correct expression for velocity 0.1

A beaker of helium is filled with triangular lattice ($u \ll R_0$) of identical vertical vortices, see Fig. 5.

B2 (0.15)

Draw the trajectories of vortices A, B, and C (located in the center).



- Trajectories are plotted correctly (0.05 for each) 0.15

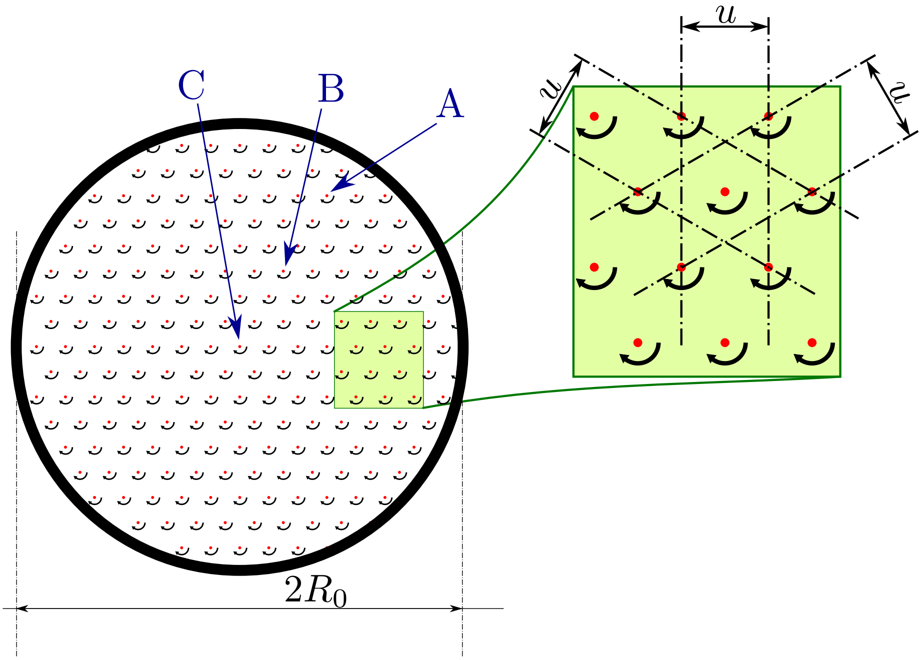


Figure 5: Triangular lattice of vortices in a beaker. The view from above.

B3 (0.4)

Find velocity $v(\vec{r})$ of a vortex positioned at $\vec{r}$.

Consider a circular path of radius $r \gg u$ around the beaker center. The circulation along this path is given by the number of vortices within it (vortex density per unit area is $(u^2\sqrt{3}/2)^{-1}$):

$$2\pi r v = 2\pi\kappa \frac{\pi r^2}{u^2\sqrt{3}/2}. \tag{4}$$

The velocity field

$$v = \frac{2\pi\kappa r}{u^2\sqrt{3}}. \tag{5}$$

- Expression for vortex density 0.2
- Correct expression for $v(r)$ 0.2

B4 (0.2)

Find the distance AB(t) between the vortices A and B at time t. Treat AB(0) as given.

This velocity pattern corresponds to the rotation of the lattice as a whole around the beaker center with angular velocity

$$\omega = \frac{2\pi\kappa}{u^2\sqrt{3}}. \tag{6}$$

AB(t) = AB(0)

- Correct answer 0.2

B5 (0.25)

Work out the "smoothed out" (omitting the lattice structure) free helium surface shape $z(\vec{r})$.

The surface shape is

$$z(r) = \frac{\omega^2 r^2}{2g} = \frac{4\pi^2\kappa^2 r^2}{6gu^4}. \tag{7}$$

- Correct answer 0.25

C. Momentum and Energy (1.75)

The long range velocity field is the major contribution to the energy of a system of vortices, it is insensitive to exact structure of the filament. The filament itself can not be properly described by the macroscopic theory and apparent singularities (infinities) are insignificant. Real physical quantities, such as the energy, of the region inside a thin tube of radius a around the filament should be neglected. Outside of this tube the density of superflow kinetic energy $\rho v^2/2$ (where $\rho = \text{const}$) is similar to the energy density of the magnetic field $B^2/(2\mu_0)$ — they are both quadratic in respective variables. This similarity facilitates calculation of the flow energy for a given system. For instance, given the inductance of a circular wire loop $L \approx \mu_0 R \log(R/a)$, where R is the loop radius and a is wire radius, we get the superfluid vortex loop energy[3]

$$U \approx 2R\rho\pi^2\kappa^2 \log(R/a). \tag{8}$$

Total fluid momentum is also determined by the long range velocity distribution. It is obtained by integration of the momentum density $\rho\vec{v}$. Again, consider

[3] This expression is also valid only if $\log R/a \gg 1$.

a flow generated by a circular vortex loop placed in xy plane. It is obvious from the symmetry considerations, that total momentum has only z component:

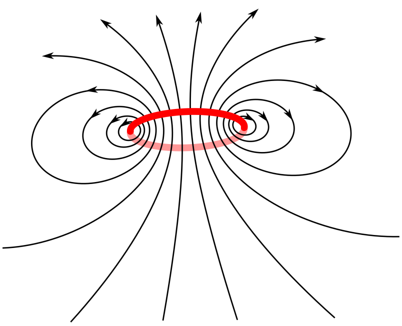
$$P = \int \rho v_z dV = \rho \int\int \underbrace{\left(\int v_z dz\right)}_{q(x,y)} dxdy \tag{9}$$

The innermost integration is in fact an integration along appropriate paths parallel to z-axis, see Fig. 6. From the circulation identity (1) it follows that

$$q(x,y) = \int\limits_{L(x,y)} \vec{v} \cdot d\vec{l}$$

is piecewise constant. Particularly, it is zero for paths passing outside the ring and $2\pi\kappa$ for paths inside it. Total momentum is therefore

$$P = \rho \cdot \pi R^2 \cdot 2\pi\kappa = 2\pi^2 \rho R^2 \kappa. \tag{10}$$



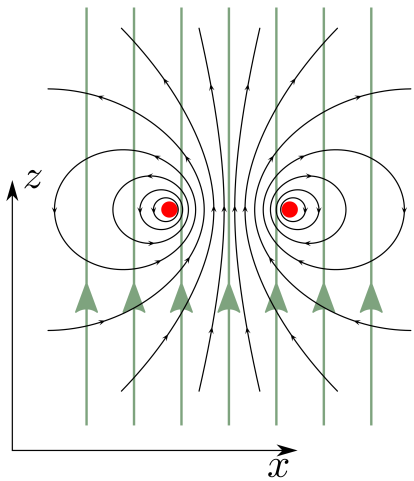


Figure 6: Velocity field of a circular vortex loop and integration paths (green) for $q(x,y)$ calculation.

C1 (0.3)

Consider a nearly rectangular vortex loop $b \times d$, $b \ll d$, Fig. 7.

Indicate the direction of its momentum $\vec{P}$. Find out the momentum magnitude.

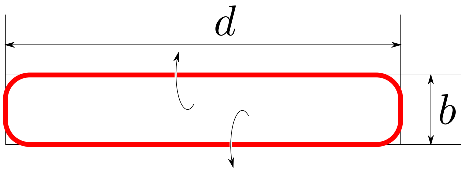
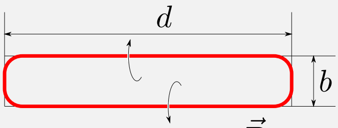


Figure 7: A nearly rectangular vortex loop, $b \ll d$



$\odot \vec{P}$ Momentum of a flat loop (see Introduction) is perpendicular to its plane and proportional to its area. For a rectangular loop the magnitude is $P = 2\pi\kappa\rho bd$.

- Correct direction of momentum 0.15
- Correct expression for momentum magnitude 0.15

C2 (0.7)

Calculate its energy U.

To produce equal magnetic and kinetic energy densities $B^2/(2\mu_0) = \rho v^2/2$, the magnetic field has to be $B = v\sqrt{\mu_0\rho} = \kappa\sqrt{\mu_0\rho}/r$. This field is generated by a current $I = 2\pi\kappa\sqrt{\rho/\mu_0}$. Energy of the wire loop can be found from the inductance $U = LI^2/2$. Inductance of a nearly rectangular wire loop:

$$L = \frac{\Phi}{I} = 2dI^{-1}\int_a^b \frac{\mu_0 I}{2\pi r}dr = \frac{\mu_0 d}{\pi}\log\frac{b}{a}. \tag{11}$$

This gives for the energy

$$U = 2\pi\kappa^2\rho d\log\frac{b}{a} \tag{12}$$

- Integration limits are a and b 0.2
- Analogy with a magnitude field is used ($U = \frac{LI^2}{2}$, $L = \frac{\Phi}{I}$) or energy is calculated as $W = \int F\,dr$, where $F = \frac{dP}{dt}$ 0.2
- Correct expression for energy 0.3

C3 (0.75)

Suppose we shift a long straight vortex filament by a distance b in x direction, see Fig. 8. How much does the fluid momentum change? Indicate the momentum change direction. The filament length (constrained by the vessel walls) is d.

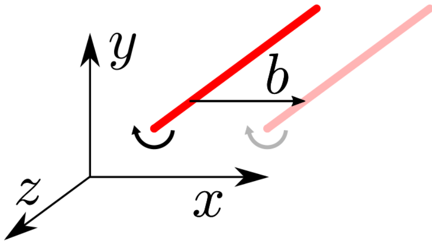
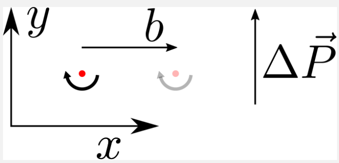


Figure 8: Momentum changes whenever the vortex shifts with respect to the fluid.



The momentum change is equal to the momentum of a long rectangular loop $P = 2\pi\kappa\rho bd$.

- The result of C1 used 0.3
- Momentum change is parallel to Y axis 0.1
- Correct direction of momentum change 0.15
- Correct expression for momentum change magnitude 0.2

Interestingly, this provides an alternative approach to find the energy of such a loop. Namely, if we slowly move one straight vortex in the velocity field of another, then we apply a force

$$F = 2\pi\kappa\rho dv = 2\pi\kappa\rho d\frac{\kappa}{r} = \frac{2\pi\kappa^2\rho d}{r}. \tag{13}$$

The work

$$W = \int_a^b \frac{2\pi\kappa^2\rho d}{r} dr = 2\pi\kappa^2\rho d \log\frac{b}{a} \tag{14}$$

has to be performed to move it from distance a to b.

D. Trapped charges (2.5)

Electrons, if injected in helium, get trapped by the vortex filaments. Here and below polarizability of helium can be neglected ($\epsilon = 1$).

D1 (0.5)

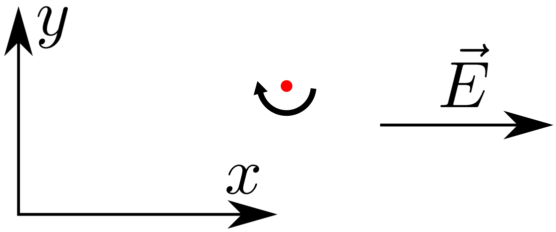
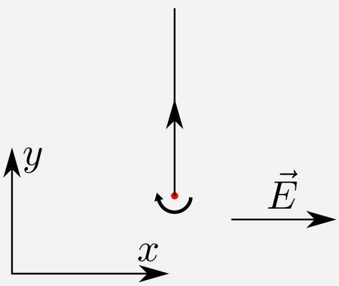


Figure 9: Straight vortex in a uniform electric field.

Consider a straight vortex charged with uniform linear density $\lambda < 0$ in a uniform electric field $\vec{E}$. Draw the vortex trajectory. Find its velocity as a function of time.



Electric force $F = E\lambda d$ moves the vortex with velocity

$$v = \frac{F}{2\pi\kappa\rho d} = \frac{E\lambda}{2\pi\kappa\rho} \tag{15}$$

perpendicular to $\vec{E}$.

- Trajectory is straight line parallel to Y axis 0.1
- Correct direction of velocity 0.2
- Correct expression for velocity magnitude 0.2

A circular vortex loop of radius R_0 initially charged with uniform linear density $\lambda < 0$ is placed in a uniform electric field $\vec{E}$ perpendicular to its plane, opposite to its momentum $\vec{P}_0$.

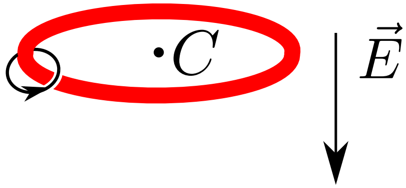
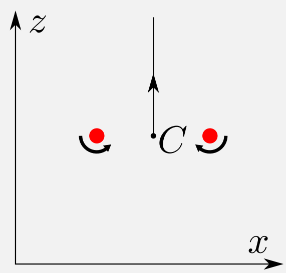


Figure 10: Vortex ring in a uniform electric field.

D2 (0.25)

Draw the trajectory of the loop center C.



- Trajectory plotted correctly 0.25

D3 (1.5)

Find its velocity $v(t)$ as a function of time.

Electric force upon the loop $F = 2\pi E R_0 \lambda$ is constant and fluid momentum linearly depends on time

$$P = P_0 + 2\pi E R_0 \lambda t = 2\pi^2 \rho R^2 \kappa. \tag{16}$$

The loop is growing and its radius is increasing with time t

$$R = \sqrt{R_0^2 + \frac{E R_0 \lambda t}{\pi \rho \kappa}}. \tag{17}$$

The loop velocity v can be easily found from a relationship between the energy change rate and the momentum change rate

$$\frac{dU}{dt} = Fv = \frac{dP}{dt} v. \tag{18}$$

This gives for the velocity

$$v = \frac{dU}{dP} \approx \frac{\kappa}{2R} \log \frac{R}{a} = \frac{\kappa \log \frac{\sqrt{R_0^2 + E R_0 \lambda t/(\pi\rho\kappa)}}{a}}{2\sqrt{R_0^2 + E R_0 \lambda t/(\pi\rho\kappa)}} \approx \frac{\kappa \log \frac{R_0}{a}}{2\sqrt{R_0^2 + E R_0 \lambda t/(\pi\rho\kappa)}}. \tag{19}$$

This means that the vortex is moving in the direction of the force but its velocity is decreasing.

- Correct expression for $R(t)$ 0.5
- Expression for $v \propto \frac{1}{R} \log(R)$ 0.5
- Correct expression for $v(t)$ 0.5

D4 (0.25)

The field is switched off at a time t^* when the velocity reaches the value $v^* = v(t^*)$. Find the loop velocity $v(t)$ at a later time $t > t^*$.

When $E = 0 \Longrightarrow P = const \Longrightarrow R = const \Longrightarrow v = const \Longrightarrow v(t) = v^*$.

- Correct expression for $v(t)$ 0.25

E. Influence of the boundaries (2.75)

Solid walls alter the velocity field created by a vortex filament, because the fluid cannot flow through them. Mathematically this means that the wall-normal velocity component vanishes at the wall surface.

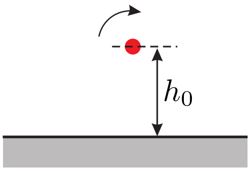


Figure 11: Straight vortex filament near a flat wall

E1 (0.5)

Draw the trajectory of a straight vortex, initially placed at a distance h_0 from a flat wall. Find its velocity as a function of time.

Well known technique of image charges (currents) in electrostatics (magnetostatics) can be directly used to solve this problem. Namely, the wall can be "substituted" with a reflected fictitious vortex on the other side of the wall. The velocity distribution of two vortices together in the upper semi-space is identical to the one produce by a single vortex above the wall. Indeed, the symmetry of the problem ensures that there is no flow through the plane of symmetry. Thus, a straight vortex line situated a distance h_0 above a flat wall with its image behave as a pair of vortices of opposite circulation a distance $2h_0$ apart. This means that the vortex moves along the wall with velocity

$$v = \frac{\kappa}{2h_0}. \tag{20}$$

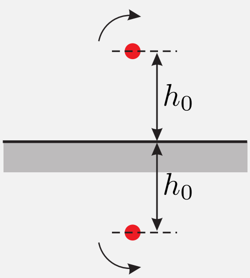


Illustration of the image method for the straight vortex filament near a flat wall

- Trajectory is plotted correctly 0.25
- Correct direction of velocity 0.1
- Correct expression for velocity magnitude 0.15

Consider a straight vortex placed in a corner at a distance h_0 from both walls.

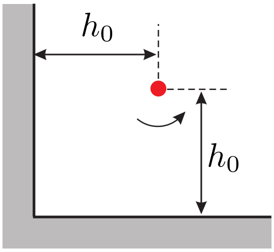


Figure 12: Straight vortex filament in a corner

E2 (0.75)

What is the initial velocity v_0 of the vortex?

The velocity of the filament is given by superposition of the velocities $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ induced by the image vortices 1, 2 and 3, respectively (see Fig. in E3 solution). One readily obtains

$$v_1 = \frac{\kappa}{2h_0}, \quad v_2 = \frac{\kappa}{2\sqrt{2}h_0}, \quad v_3 = \frac{\kappa}{2h_0}.$$

The modulus of the filament velocity at the initial moment is

$$v_0 = |\vec{v}_1 + \vec{v}_2 + \vec{v}_3| = \sqrt{2}v_1 - v_2 = \boxed{\frac{\kappa}{2\sqrt{2}h_0}}$$

- Ideas of using superposition principal and technique of image charges 0.25
- Correct direction of initial velocity 0.2
- Correct expression for initial velocity magnitude 0.3

E3 (0.5)

Draw the trajectory of the vortex.

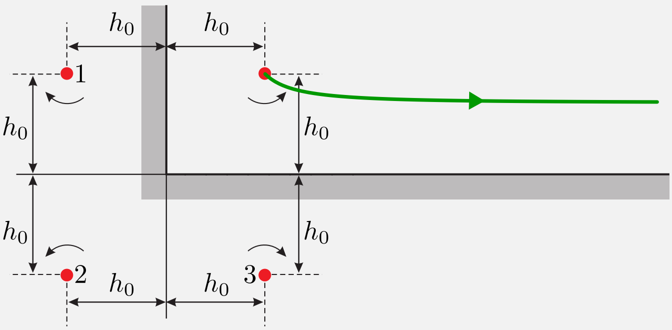


Image vortices in the corner.

- The trajectory has correct form and asymptote 0.5

E4 (1.0)

What is the velocity of the vortex v_∞ after very long time?

Energy for the system of vortices is proportional to

$$U_{\text{tot}} \propto \log \frac{\sqrt{x^2+y^2}}{a} - \log \frac{x}{a} - \log \frac{x}{a}. \tag{21}$$

The energy conservation implies that

$$C = \frac{x^2+y^2}{x^2y^2} = \frac{2}{h_0^2} \tag{22}$$

is constant along the trajectory. After very long time $y \to h_0/\sqrt{2}$ and the vortex velocity is

$$v_\infty = \frac{\kappa}{h_0\sqrt{2}}. \tag{23}$$

- Correct expression for velocity after very long time 1.0

F. Charges + Walls (1.0)

The vortices in this section are charged with uniform linear density $\lambda < 0$.

F1 (0.5)

Draw the trajectory of a straight vortex initially placed at a distance h_0 to a dielectric wall with charge density $\sigma < 0$. Find its velocity as a function of time.

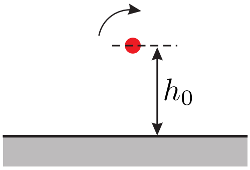
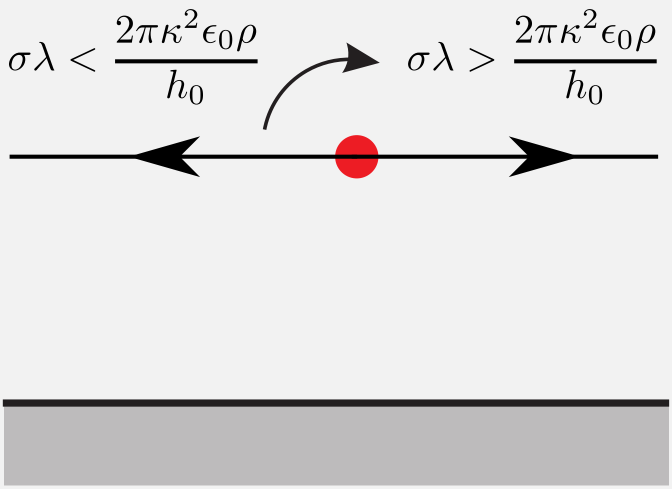


Figure 13: Straight vortex near a charged wall



The vortex velocity in electric field $E = \sigma/(2\varepsilon_0)$ is summed up with the fluid velocity generated by the image vortex

$$v = \frac{\kappa}{2h_0} - \frac{\sigma\lambda}{4\pi\varepsilon_0\kappa\rho} \tag{24}$$

and parallel to the wall. Its direction depends on the competition of the two terms above.

- Trajectory is plotted correctly 0.25
- Correct expression for velocity 0.25

F2 (0.5)

Draw the trajectory of a straight vortex initially placed at a distance h_0 to a conducting wall. Find its velocity as a function of time.

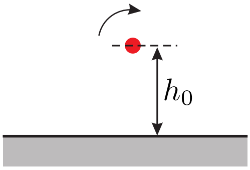
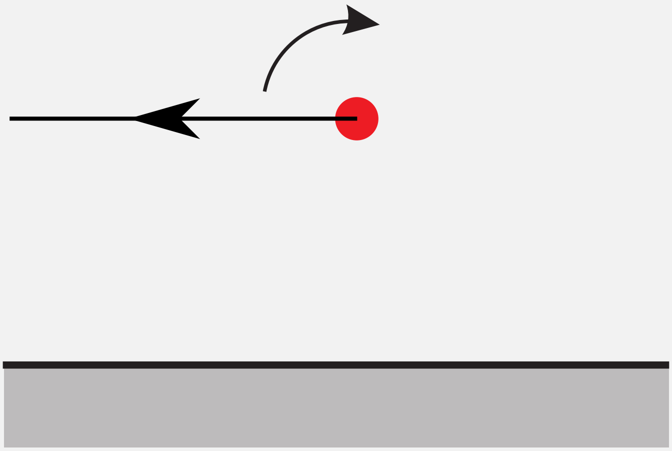


Figure 14: Straight vortex near a conducting wall



The conducting wall produces electric field identical to the field of an image filament with opposite charge density $E = -\lambda/(4\pi\varepsilon_0 h_0)$. The velocity is then

$$v = \frac{\lambda^2}{8\pi^2\varepsilon_0\kappa\rho h_0} + \frac{\kappa}{2h_0}. \tag{25}$$

- Trajectory is plotted correctly 0.25
- Correct expression for velocity 0.25