Question 1

The fractional quantum Hall effect (FQHE) was discovered by D. C. Tsui and H. Stormer at Bell Labs in 1981. In the experiment electrons were confined in two dimensions on the GaAs side by the interface potential of a GaAs/AlGaAs heterojunction fabricated by A. C. Gossard (here we neglect the thickness of the two-dimensional electron layer). A strong uniform magnetic field B was applied perpendicular to the two-dimensional electron system. As illustrated in Figure 1, when a current I was passing through the sample, the voltage V_{H} across the current path exhibited an unexpected quantized plateau (corresponding to a Hall resistance $R_{\mathrm{H}} = 3h/e^2$) at sufficiently low temperatures. The appearance of the plateau would imply the presence of fractionally charged quasiparticles in the system, which we analyze below. For simplicity, we neglect the scattering of the electrons by random potential, as well as the electron spin.

(a) In a classical model, two-dimensional electrons behave like charged billiard balls on a table. In the GaAs/AlGaAs sample, however, the mass of the electrons is reduced to an effective mass m^* due to their interaction with ions.

(i) (**2 point**) Write down the equation of motion of an electron in perpendicular electric field $\vec{E} = -E_y\hat{y}$ and magnetic field $\vec{B} = B\hat{z}$.

(ii) (**1 point**) Determine the velocity v_{s} of the electrons in the stationary case.

(iii) (**1 point**) Which direction is the velocity pointing at?

(b) (**2 points**) The Hall resistance is defined as $R_{\mathrm{H}} = V_{\mathrm{H}}/I$. In the classical model, find R_{H} as a function of the number of the electrons N and the magnetic flux $\phi = BA = BWL$, where A is the area of the sample, and W and L the effective width and length of the sample, respectively.

(c) (**2 points**) We know that electrons move in circular orbits in the magnetic field. In the quantum mechanical picture, the impinging magnetic field B could be viewed as creating tiny whirlpools, so-called vortices, in the sea of electrons—one whirlpool for each flux quantum h/e of the magnetic field, where h is the Planck's constant and e the elementary charge of an electron. For the case of $R_{\mathrm{H}} = 3h/e^2$, which was discovered by Tsui and Stormer, derive the ratio of the number of the electrons N to the number of the flux quanta N_ϕ, known as the filling factor ν.

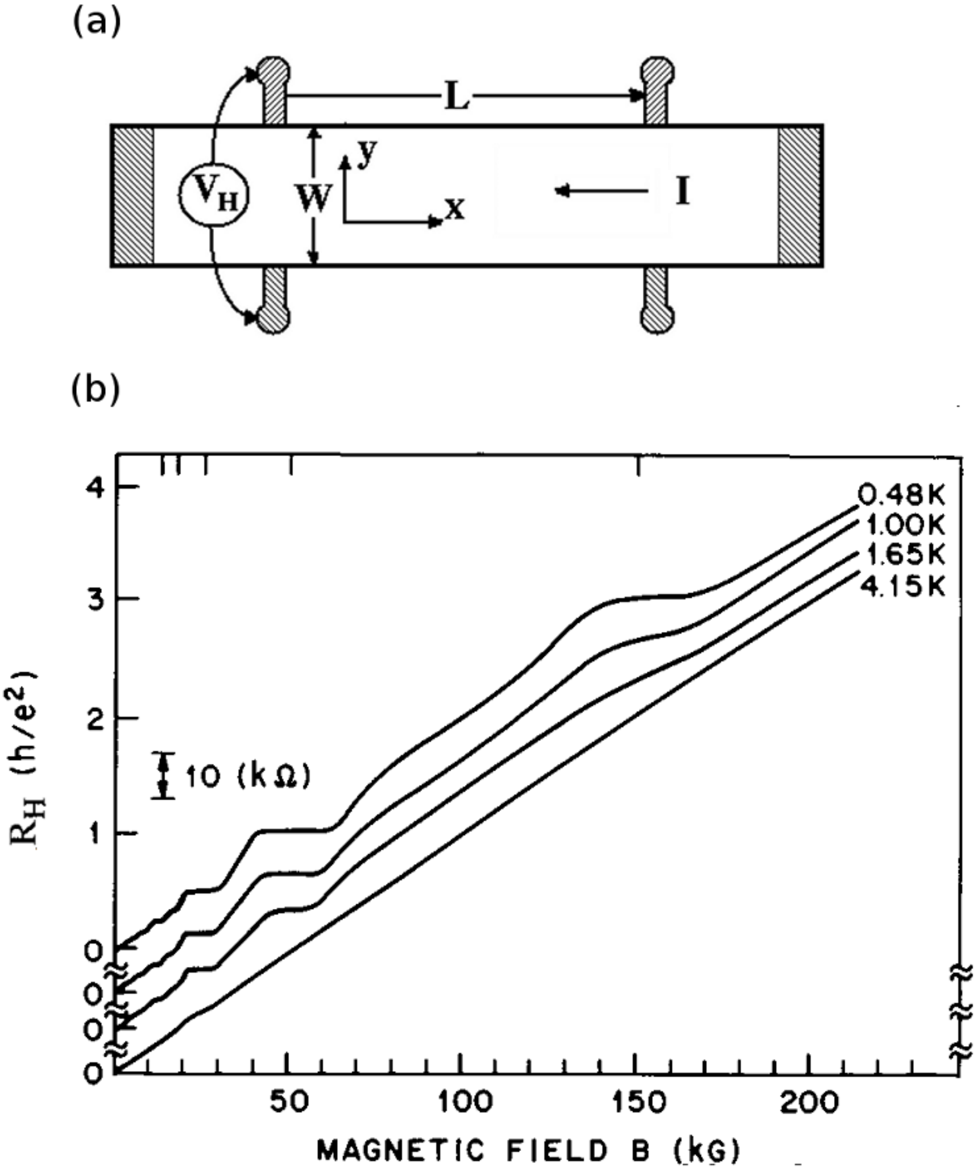

Figure 1: (a) Sketch of the experimental setup for the observation of the FQHE. As indicated, a current I is passing through a two-dimensional electron system in the longitudinal direction with an effective length L. The Hall voltage V_{H} is measured in the transverse direction with an effective width W. In addition, a uniform magnetic field B is applied perpendicular to the plane. The direction of the current is given for illustrative purpose only, which may not be correct. (b) Hall resistance R_{H} versus B at four different temperatures (curves shifted for clarity) in the original publication on the FQHE. The features at $R_{\mathrm{H}} = 3h/e^2$ are due to the FQHE.

(d) (**2 points**) It turns out that binding an integer number of vortices ($n > 1$) with each electron generates a bigger surrounding whirlpool, hence pushes away all other electrons. Therefore, the system can considerably reduce its electrostatic

Coulomb energy at the corresponding filling factor. Determine the scaling exponent α of the amount of energy gain for each electron $\Delta U(B) \propto B^{\alpha}$.

(e) (**2 points**) As the magnetic field deviates from the exact filling $\nu = 1/n$ to a higher field, more vortices (whirlpools in the electron sea) are being created. They are not bound to electrons and behave like particles carrying effectively positive charges, hence known as quasiholes, compared to the negatively charged electrons. The amount of charge deficit in any of these quasiholes amounts to exactly $1/n$ of an electronic charge. An analogous argument can be made for magnetic fields slightly below ν and the creation of quasielectrons of negative charge $e^* = -e/n$. At the quantized Hall plateau of $R_\mathrm{H} = 3h/e^2$, calculate the amount of change in B that corresponds to the introduction of exactly one fractionally charged quasihole. (When their density is low, the quasiparticles are confined by the random potential generated by impurities and imperfections, hence the Hall resistance remains quantized for a finite range of B.)

(f) In Tsui *et al.* experiment,

the magnetic field corresponding to the center of the quantized Hall plateau

$R_\mathrm{H} = 3h/e^2$, $B_{1/3} = 15$ Tesla,

the effective mass of an electron in GaAs, $m^* = 0.067\, m_e$,
the electron mass, $m_e = 9.1 \times 10^{-31}$ kg,
Coulomb's constant, $k = 9.0 \times 10^9\ \mathrm{N \cdot m^2/C^2}$,
the vacuum permittivity, $\varepsilon_0 = 1/4\pi k = 8.854 \times 10^{-12}$ F/m,
the relative permittivity (the ratio of the permittivity of a substance to the vacuum permittivity) of GaAs, $\varepsilon_r = 13$,
the elementary charge, $e = 1.6 \times 10^{-19}$ C,
Planck's constant, $h = 6.626 \times 10^{-34}\ \mathrm{J \cdot s}$, and
Boltzmann's constant, $k_\mathrm{B} = 1.38 \times 10^{-23}$ J/K.

In our analysis, we have neglected several factors, whose corresponding energy scales, compared to $\Delta U(B)$ discussed in (d), are either too large to excite or too small to be relevant.

(i) (**1 point**) Calculate the thermal energy E_th at temperature $T = 1.0$ K.

(ii) (**2 point**) The electrons spatially confined in the whirlpools (or vortices) have a large kinetic energy. Using the uncertainty relation, estimate the order of magnitude of the kinetic energy. (This amount would also be the additional energy penalty if we put two electrons in the same whirlpool, instead of in two separate whirlpools, due to Pauli exclusion principle.)

(g) There are also a series of plateaus at $R_\mathrm{H} = h/ie^2$, where $i = 1, 2, 3, \ldots$ in Tsui *et al.* experiment, as shown in Figure 1(b). These plateaus, known as the integer quantum Hall effect (IQHE), were reported previously by K. von Klitzing in 1980. Repeating (c)-(f) for the integer plateaus, one realizes that the novelty of the FQHE lies critically in the existence of fractionally charged quasiparticles. R.

de-Picciotto *et al.* and L. Saminadayar *et al.* independently reported the observation of fractional charges at the $\nu = 1/3$ filling in 1997. In the experiments, they measured the noise in the charge current across a narrow constriction, the so-called quantum point contact (QPC). In a simple statistical model, carriers with discrete charge e^* tunnel across the QPC and generate charge current I_B (on top of a trivial background). The number of the carriers n_τ arriving at the electrode during a sufficiently small time interval τ obeys Poisson probability distribution with parameter λ

$$\mathrm{P}(n_\tau = k) = \frac{\lambda^k \mathrm{e}^{-\lambda}}{k!}$$

where $k!$ is the factorial of k. You may need the following summation

$$\mathrm{e}^\lambda = \sum_{\mathrm{k}=0}^{\infty} \frac{\lambda^k}{k!}$$

(i) (**2 point**) Determine the charge current I_B, which measures total charge per unit of time, in terms of λ and τ.

(ii) (**2 points**) Current noise is defined as the charge fluctuations per unit of time. One can analyze the noise by measuring the mean square deviation of the number of current-carrying charges. Determine the current noise S_I due to the discreteness of the current-carrying charges in terms of λ and τ.

(iii) (**1 point**) Calculate the noise-to-current ratio S_I/I_B, which was verified by R. de-Picciotto *et al.* and L. Saminadayar *et al.* in 1997. (One year later, Tsui and Stormer shared the Nobel Prize in Physics with R. B. Laughlin, who proposed an elegant ansatz for the ground state wave function at $\nu = 1/3$.)