Optical trap of neutral atoms (12 points)

Optical traps are versatile tools to create ultracold atom systems that nowadays play very important role in quantum physics and are believed to have highly nontrivial applications in technology as well as in quantum measurements. By shining laser beam onto an assembly of neutral atoms, we are able to capture and cool these atoms. When atoms are cooled to near absolute zero temperature, they reveal the whole fascinating quantum behavior, in particular their Bose-Einstein condensation (BEC).

In this problem you will study basic concepts of optical trap of neutral atoms and one of the signatures to recognize the BEC in experiments on sodium atoms.

An neutral sodium atom can be well described as a core with positive charge *e* surrounded by homogenous electron cloud with negative charge *-e.* The mass of the core is much larger than the mass of the electron cloud. In the absence of an external electric field, the core and the cloud centers coincide. The electric field of laser beam interacts with the positive core as well as the electron cloud of the atom, so an electric dipole is induced. In turn, this induced dipole will interact with the electric field of laser beam and thereby gives rise to a dipole potential energy of the atom. One says that the atom feels an optical potential. The later depends on the intensity profile $I(\vec{r})$ as well as the frequency of the laser beam in use. By choosing appropriate laser intensity and frequency, one may form a trap-like potential well to confine the neutral atoms.

We start off by considering the polarization of a neutral atom that is placed in a uniform external electric field $\vec{E}_0 = E_0\hat{u}$, where $\hat{u}$ is a unit vector and E_0 is the field magnitude. Then, a *dipole moment* $\vec{p}_0 = e\ell\hat{u} = \alpha E_0\hat{u}$ is induced. Here, ℓ is distance between the negative and positive charge centers, and α is called *polarizability.*

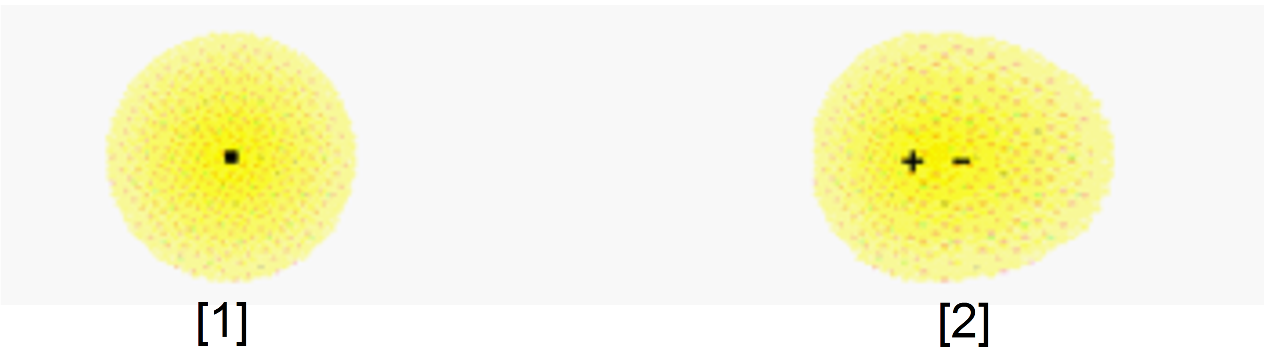

Figure 1. Electron cloud distribution. [1] Spherical distribution of electron cloud about the atomic core; [2] Shifted electron cloud (separation of **+** and **-** within the atom) in an electric field.

1 (1.5 points)

Initially the external field is turned off. Then the field magnitude is increased from zero to E_0 very slowly so that the electric field can be considered effectively time-independent in this question. The instantaneous value of the external field is denoted by $\vec{E} = E\hat{u}$,

1.1	Find the instantaneous power absorbed by the atom from the external field in terms of $\vec{E}$ and $\dot{\vec{p}}$, where $\dot{\vec{p}}$ is the rate change of the induced dipole moment.	0.75pt

1.2	Find the total work done by the external field on the atom when the electric field increased from zero to $E = E_0$, hence deduce an expression for the induced dipole potential energy $U_{induced}$ in terms of $\vec{E}_o$ and $\vec{p}_o$.	0.75pt

Note that when the external electric field is turned off, the electron cloud oscillates with the natural frequency ω_0 due to its inertia and the Coulomb restoring force.

2 (1.0 point)

In the following we will study the case where the neutral atoms are placed in an external field of a laser that varies in time and space as $\vec{E}(\vec{r},t) = \hat{u}.E_0(\vec{r})\cos\omega t$. The induced dipole moments $\vec{p}$ will oscillate with the driving laser field frequency ω. It is well-known that an oscillating dipole itself emits electromagnetic radiation. By doing so, electron receives some recoil momentum that causes an electromagnetic friction resulting in a phase shift between the applied electric field and the induced dipole moment. Therefore, the induced dipole moment takes the form $\vec{p}(\vec{r},t) = \hat{u}E_0(\vec{r})\alpha(\omega)\cos[\omega t + \varphi(\omega)]$. Here, both the polarizability α and the phase shift φ depend on the driving frequency ω . Due to the oscillating nature, all physical quantities of our interest reveal themselves only via the corresponding time-averaged values over a period $2\pi/\omega$ of the laser field. Time-averaged value of a periodically varying quantity is defined as $\langle f(t)\rangle = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} f(t)\,dt$. Hereafter, the notation $\langle ...\rangle$ means time-average of the enclosed quantity.

Laser intensity $I(\vec{r})$ is related to amplitude of the laser electric field E_0 as $I(\vec{r}) = \frac{\varepsilon_0 c E_0^2(\vec{r})}{2}$, where ϵ_0 is the permittivity of free space and c is the speed of light.

2.1	Find the induced dipole potential energy $U_{dip}(\vec{r}) = \langle U_{induced}(\vec{r},t)\rangle$ in term of $\alpha,\ \varphi,\ \varepsilon_0,\ c$, and $I(\vec{r})$.	1.0pt

3 (1.0 point)

Besides capturing neutral atoms in the trap via the induced dipole potential energy, the oscillating electric field may cause the scattering force on atoms that arises from absorption and emission of light. The light scattering processes lead either to heating or to losses of atoms from the trap and may be characterized by the scattering rate, that is related to the number of photons scattered by an atom in unit time and is defined by $\Gamma_{sc}(\vec{r}) = \dfrac{\langle P_{abs}(\vec{r})\rangle}{\hbar\omega}$. Here, $\langle P_{abs}(\vec{r})\rangle$ is the time-averaged power absorbed from the laser field, and $\hbar\omega$ is the photon energy $(\hbar = h/2\pi)$.

3.1	Find the scattering rate $\Gamma_{sc}(\vec{r})$ in term of $\alpha,\ \varphi,\ \varepsilon_0,\ c,\ I(\vec{r}), \hbar$ and ω.	1.0pt

4 (2.0 points)

Both quantities U_{dip} and $\Gamma_{sc}(\vec{r})$ depend on the polarizability α. In order to calculate the polarizability α, we will adopt the one dimensional oscillator model under the presence of an electric field $\vec{E}(t) = \hat{u}E_0\cos\omega t$. Call *Ox* the axis parallel to the unit vector $\hat{u}$. In this model motion of the electron is determined by three forces:

i) The restoring force $-m_e\omega_0^2 x\cdot\hat{u}$ that describes the free oscillation with the natural frequency ω_0 corresponding to the atomic optical transition frequency. We use *x* to denote the displacement of the negative charge center from the positive one, which is assumed to be at rest.

ii) The driving force of the laser field $-eE_0 cos\omega t.\hat{u}$

iii) The damping force $-m_e\gamma_\omega\dot{x}.\hat{u}$ that originates from the radiation of the accelerating charge, and is characterized by the frequency-dependent damping rate γ_ω.

Therefore, the equation of motion of the electron is given as $\ddot{x}+\gamma_\omega\dot{x}+\omega_0^2x=\frac{-eE_0\cos\omega t}{m_e}$. The solution to this equation is $x=x_0\cos(\omega t+\varphi)$. Here x_0 and φ are to be determined.

4.1	Find the polarizability α in term of γ_ω, e, m_e, ω_0, and ω.	2.0pt

5 (1.0 point)

In fact the energy damping rate γ_ω is independent of the electron orbits. Therefore we will adopt another simple model where the electron cloud center performs a circular motion in the absence of the laser field but with the frequency ω and speed v. Being accelerated, the electron radiates an electromagnetic wave with power given by the Larmor formula $P_L=\frac{1}{6\pi\varepsilon_0}\frac{e^2a^2}{c^3}$ with a denoting acceleration. The damping force is supposed to be related to the damping rate γ_ω as $F_d=-m_e\gamma_\omega v$. We also assume that the total energy of the electron is large compared with the energy loss per cycle.

5.1	Find the energy damping rate γ_ω in term of e, ϵ_0, c, m_e, and ω.	1.0pt

6 (0.5 point)

When the driving frequency ω is close to the natural frequency ω_0, the the polarizability gets larger, leading to a larger value of the dipole potential as well as increased scattering rate. Therefore, by considering the ratio $U_{dip}(\vec{r})/\hbar\Gamma_{sc}(\vec{r})$, one can find an appropriate laser frequency to reduce the scattering rate while maintaining a reasonably deep trapping potential.

6.1	Introducing the damping rate at $\omega=\omega_0$, as $\gamma\equiv\gamma_{\omega_0}$, find the ratio $U_{dip}(\vec{r})/\hbar\Gamma_{sc}(\vec{r})$ in terms of ω, ω_0, and γ.	0.5pt

7 (1.5 points)

From the above result we can see that it is possible to simultaneously achieve a deep trapping potential and low heating rates by choosing the laser frequency ω being not too close to the atomic optical transition ω_0, as well as high laser intensity. Because the scattering rate $\Gamma_{sc}(\vec{r})$ is positive, and from the above obtained ratio $U_{dip}(\vec{r})/\hbar\Gamma_{sc}(\vec{r})$one can see if $\omega<\omega_0$, the dipole potential is negative and the atoms are captured in a focused region of laser beam with maximum intensity. Once atoms are captured in the trap, by reducing the trapping well depth to remove high energy atoms, one may cool the confined atom gas to ultracold temperatures, enabling formation of BEC. A breakthrough progress in BEC physics had been achieved with sodium atoms $^{23}\mathrm{Na}$ in the late nineties (D. M. Stamper-Kurn et al., Phys.Rev.Lett. 80, 2027 (1998)).

The physics of BEC can be understood as follows. In nature, there are two kinds of particles: bosons with integer spin and fermions with half integer spin. Two identical fermions cannot exist in the same quantum state. In contrast, multiple bosons are not forbidden to occupy one quantum state: at ultralow temperatures a large fraction of bosons can condensate into the state with lowest possible energy and form a condensate cloud (condensate bosons), while the rest bosons are in the excited state with higher energy (noncondensate or thermal bosons). Let us analyse a practical example of a dilute gas of sodium atoms, which are bosons, confined in the optical trap created by a Gaussian laser beam (Fig 2a). The laser beam has the wavelength λ corresponding to the frequency ω (with $\omega<\omega_0$). The beam propagates along the z-axis with the intensity profile $I(\rho,z)=\frac{2P}{\pi D(z)^2}\exp\left(-\frac{2\rho^2}{D(z)^2}\right)$, where $\rho=\sqrt{x^2+y^2}$ and the waist size is $D(z) = D_0\sqrt{1+z^2/z_R^2}$ with $z_R = \pi D_0^2/\lambda$ denoting the Rayleigh length. The total laser power *P* and the beam waist parameter D_0 determine the parameters of the optical trapping potential, one of which is the potential depth U_{depth}. The later is defined by the absolute value of the local minimum of the potential energy, taking zero reference at infinity (Fig 2b).

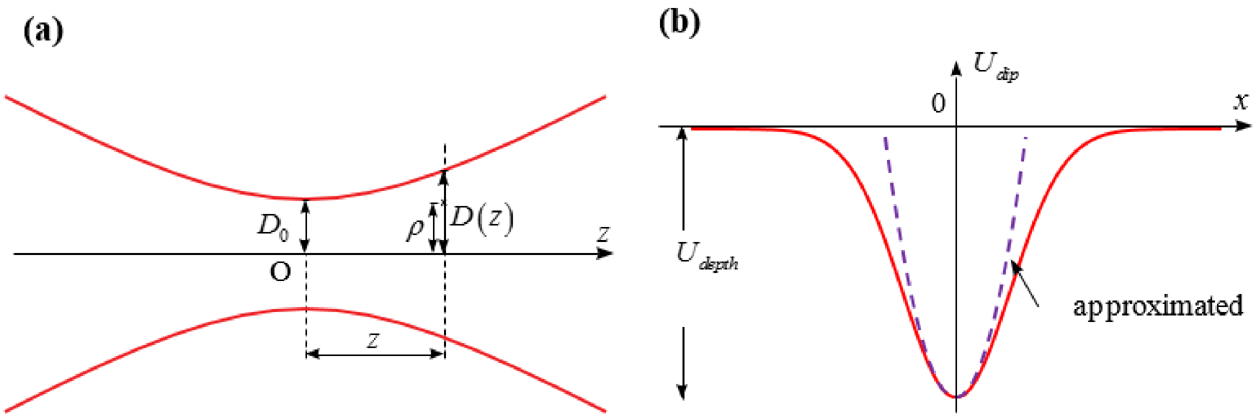

Figure 2. (a) Gaussian beam. The envelope represents the beam waist $D(z)$ at the fix plane $z = const$. (Adopted from wikipedia); (b) Illustration of optical trap along x-axis created by a Gaussian beam with $\omega < \omega_0$. The dash line corresponds to a harmonic approximation near the trap bottom.

7.1	Find the expression for the dipole potential depth U_{depth} in terms of $c, \omega, \omega_0, \gamma, P$, and D_0.	0.5pt

7.2	Given laser power $P = 4$ mW, laser wavelength $\lambda = 985$ nm, $D_0 = 6\,\mu$m, and natural wave length for sodium $\lambda_0 = 589$ nm. Evaluate the potential depth U_{depth}, expressing your answer as an equivalent temperature T_0, at which thermal energy of the non-trapped atom is equal to the trap depth.	1.0pt

8 (0.5 point)

When the cloud temperature T is much smaller than equivalent temperature T_0, the optical potential can be well approximated by a cylindrically symmetric harmonic potential $U_{dip}(\rho, z) = -U_{depth} + \frac{1}{2}m\Omega_\rho^2\rho^2 + \frac{1}{2}m\Omega_z^2 z^2$, where *m* is the mass of sodium atom and Ω_ρ, Ω_z are oscillation frequencies in the corresponding directions.

8.1	Find the expression for Ω_ρ, Ω_z in terms of T_0, m, D_0, z_R and k_B. Here k_B is the Boltzmann constant.	0.5pt

Recall that at ultralow temperatures, the sodium atom cloud consists of condensate atoms and thermal atoms. Condensate bosons behave according to the uncertainty principle that can be used for estimating the spatial size or the momentum distribution of the cloud. On the other hand, thermal bosons are described by classical physics, in particular, they obey the Maxwell-Boltzmann distribution law.

We estimate the size of the condensate cloud, that is the mean distance of the condensate sodium atoms from the trap center. Moving inside this cloud, each condensate atom has potential energy as well as kinetic energy. The potential energy is a monotonically increasing function of the cloud size, and the particle tries to reduce it to reach the lowest energy level. On the other hand, as the cloud size decreases, the uncertainty principle requires an increase in the particle momentum, that results in an increase of kinetic energy. The particle therefore finds an optimal cloud size to balance the two opposite tendencies of the two different energy contributions.

9 (1.0 point)

For simplicity, let us consider the simplest case of one dimensional trap potential $U(z) = const + \frac{1}{2}m\Omega_z^2 z^2$.

9.1	Estimate the size z_0 of the condensate fraction in terms of $m,\ \hbar,\ \Omega_z$.	0.5pt

9.2	Find the expression for E_{min} - the lowest energy level, in terms of $\hbar,\ \Omega_z$.	0.25pt

9.3	Find the average particle velocity ν_0 in terms of $m,\ \hbar,\ \Omega_z$.	0.25pt

In what follows we will figure out how to differentiate the condensate cloud from the thermal one by switching off the confining trap. It is neccesary to capture the image of the cloud density profile.

The thermal gas will show an isotropic Maxwell velocity distribution even if the trap is anisotropic. In contrast, the velocity distribution of a BEC is anisotropic. More precisely, the BEC expands faster along the axis of strong confinement than along the axis of weak confinement. The expansion predominantly occurs in the radial direction, and the initially cigar-shaped condensate becomes pancake-shaped. Therefore the density profile after a long time of flight will be anisotropic and inverted with respect to the shape of the cloud in the trap.

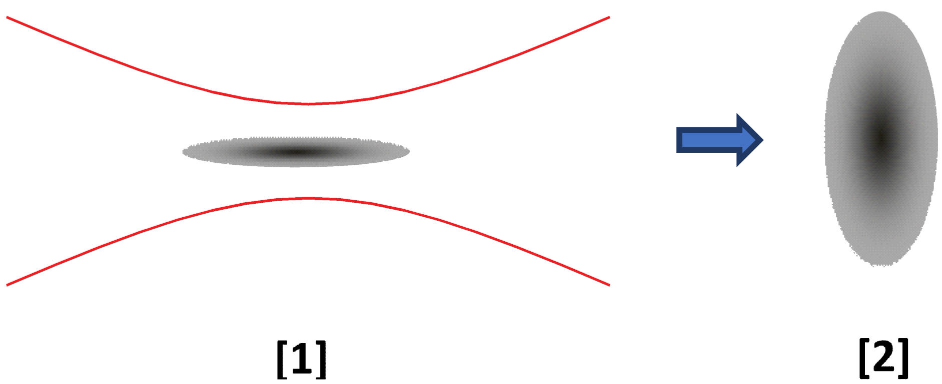

Figure 3. Cloud shape. [1] Before switching off the trap; [2] A very long time after switching off the trap.

10 (2.0 points)

Now we extend the previous results to the three-dimensional potential which is the case of the optical trap in a Gaussian laser beam.

10.1	Find the aspect ratio $\frac{z_0}{\rho_0}$ in terms of Ω_ρ, Ω_z, where z_0 and ρ_0 are the initial sizes of the condensate cloud.	0.5pt
10.2	When the trap is turned off, the condensate will be expanding in different directions with different initial velocities ν_ρ and ν_z. Determine the ratio $\frac{\nu_\rho}{\nu_z}$ in terms of Ω_ρ, Ω_z.	0.5pt
10.3	Assuming that the velocities of the cloud expansion remain unchanged during the expansion, find aspect ratio of the condensate cloud after a long period of time $\frac{z_L}{\rho_L}$ when the cloud size is much greater than its initial size, that is $z_L \gg z_0$ and $\rho_L \gg \rho_0$.	0.5pt
10.4	Same as question 10.3., find the aspect ratio of the thermal cloud after a long period of time $\frac{z_{T,L}}{\rho_{T,L}}$ when the cloud size is much greater than its initial size, that is $z_{T,L} \gg z_0$ and $\rho_{T,L} \gg \rho_0$.	0.5pt