Evolution of Supermassive Black Holes Binary
Solution

A. Dynamical Friction

A1. The deflection angle is defined from: $\tan\alpha \approx \alpha = \frac{p_y}{p_x}$, assuming that $\alpha \ll 1$. One can find $p_y = \int F_y\, dt$, and according to Newton's gravity law

$$F_y = \frac{GMm}{b^2}\cos^3\varphi$$

Changing the variable $dt = \frac{bd\varphi}{v\cos^2\varphi}$ we have

$$p_y = \frac{GM}{bv}\int_{-\pi/2}^{\pi/2}\cos\varphi\, d\varphi = \frac{2GMm}{bv}.$$

Here we assume that the body moves along the stright line, due to $\alpha \ll 1$, see Fig 1. So $\alpha = \frac{p_y}{p} = \frac{2GM}{bv^2} = \frac{2b_1}{b}$.

A2. During the transit of a massive body, star's energy remains constant: $p_x^2 + p_y^2 = \text{const}$. Hence

$$(p - \Delta p_x)^2 + p_y^2 = p^2.$$

We know that $p_y \ll p$, so the momentum change along the x-axis:

$$\Delta p_x = -\frac{p_y^2}{2p} = -\frac{\alpha^2}{2}p = -\frac{2G^2M^2m}{b^2v^3}.$$

A3. To calculate net force we might integrate over stars with different impact parameters. The number of stars' transits during the time Δt equals $\Delta N = 2\pi bvn\, db\, \Delta t$, so force, decelerating the object along the x-axis,

(1) $$F_{DF} = \frac{1}{\Delta t}\int \Delta p_x\, dN = -4\pi G^2M^2\frac{nm}{v^2}\int_{b_{min}}^{b_{max}}\frac{db}{b} = -4\pi G^2M^2\frac{\rho}{v^2}\log\Lambda$$

The above formulas are true only for $b > b_1$, so the lower integration limit is $b_{min} = b_1$, and the upper limit is determined by the galaxy size $b_{max} = R$. So we have

(2) $$F_{DF} = -4\pi G^2M^2\frac{\rho}{v^2}\log\Lambda$$

where $\Lambda = R/b_1$.

A4. We calculate: $b_1 = \frac{GM}{v^2} = 11$ pc, $\log\Lambda = 7.6$.

B. Gravitational slingshot

B1. From the second Newton's law

$$\frac{mv^2}{2} = \frac{GMm}{4a^2},$$

and we have for the orbital velocity $v_{bin} = \sqrt{\frac{GM}{4a}}$. The system energy is

$$E = E_{\text{kin}} + U = 2 \cdot \frac{Mv^2}{2} - \frac{GM^2}{2a} = -\frac{GM^2}{4a}.$$

B2. From angular momentum conservation law

$$b\sigma = r_m v_0,$$

express v_0. Write down the energy conservation law

$$\frac{\sigma^2}{2} = \frac{v_0^2}{2} - \frac{GM_2}{r_m}$$

and derive $b = r_m\sqrt{1 + \frac{2GM_2}{\sigma^2 r_m}}$.

B3. To estimate the time between collisions let us use an analogy with the gas. As known from the molecular kinetic theory, given that molecules have radii r, thermal velocities v, and the molecular concentration is n, time t between collisions of one molecule with the others can be estimated from the relation $\pi r^2 vtn = 1$. In our problem b_{max} stands in place of the molecule radius, therefore for estimation it can be written $\Delta t = \frac{1}{\sigma b_{max}^2 n}$.

Estimate the maximal impact parameter b_{max}, corresponding to the star collision with the binary system. The star should reach the distance of a to the binary system to collide. When the star is at large distances from the SBH binary, it interacts with it as with a point object of mass $M_2 = 2M$. From the results of B.2, assuming $r_m = a$, we obtain $b_{max} = a\sqrt{1 + \frac{4GM}{\sigma^2 a}}$. Taking into account that $\sigma \ll \frac{GM}{a}$, simplify: $b_{max} = \frac{2}{\sigma}\sqrt{GMa}$, so we have

$$\Delta t = \frac{m\sigma}{GM\rho a}.$$

B4. During the one act of gravitational slingshot, star energy increases at average by $\Delta E_{star} = \frac{mv_{bin}^2}{2} - \frac{m\sigma^2}{2}$.
So after the one collision the energy of the binary system decreases by the same magnitude. Taking into account that $\sigma \ll v_{bin}$, we derive $\Delta E = -\frac{m}{2}v_{bin}^2$.

Average binary system energy loss rate equals $\frac{dE}{dt} = \frac{\Delta E}{\Delta t} = -\frac{G^2M^2\rho}{4\sigma}$. In respect to $\frac{dE}{dt} = \frac{GM^2}{4a^2}$, orbit radius variation rate can be estimated as

$$\frac{da}{dt} = -\frac{G\rho a^2}{\sigma}. \quad (3)$$

B5. Obtained equation can be easily integrated

$$\frac{da}{a^2} = -\frac{G\rho}{\sigma}dt. \quad (4)$$

To reduce the radius twice it takes time $T_{SS} = \frac{\sigma}{G\rho a_1} = 0.0048\,\text{Gy}$.

C. Emission of gravitational waves

C1. Using previous results (B) one can obtain:

$$\frac{dE}{dt} = \frac{d}{dt}\left(-\frac{GM^2}{4a}\right) = \frac{GM^2}{4a^2}\frac{da}{dt}; \tag{5}$$

This leads to the differential equation:

$$\frac{GM^2}{4a^2}\frac{da}{dt} = -\frac{1024}{5}\cdot\frac{G\omega^6 M^2 a^4}{c^5}, \tag{6}$$

where angular velocity is known from part B: $\omega = \sqrt{\frac{GM}{4a^3}}$.
The desirable result is:

$$\frac{da}{dt} = -\frac{G^3M^3}{c^5a^3} \tag{7}$$

C2. Solving the equation one can obtain:

$$a^3\frac{da}{dt} = -\frac{G^3M^3}{c^5} \Rightarrow \frac{a_2^4}{4} \approx \frac{256}{5}\cdot\frac{G^3M^3}{c^5}\cdot T_{GW}; \tag{8}$$

The final result for time:

$$T_{GW} = \frac{5}{1024}\cdot\frac{a_2^4c^5}{G^3M^3}. \tag{9}$$

C3. Solving the previous equation:

$$a_H = \sqrt[4]{\frac{1024}{5}\cdot\frac{t_HG^3M^3}{c^5}} \approx 0.098\ pc. \tag{10}$$

D. Full evolution

D1. The galaxy is spherically symmetric, so mass enclosed within a sphere of radius r equals $m(r) = \int_0^r 4\pi x^2\rho(x)\,dx = \frac{\sigma^2 r}{G}$. Thus the free fall acceleration of the body equals in the gravitational field of stars is $g(r) = \frac{Gm(r)}{r^2} = \frac{\sigma^2}{r}$. Therefore the body velocity is determined by relation $\frac{v^2}{r} = g = \frac{\sigma^2}{r}$, which means $v = \sigma =$const.

D2. The energy of SBH in this gravitational field is

$$E = \frac{M\sigma^2}{2} + U$$

and $\frac{dU}{da} = g(r)M = \frac{M\sigma^2}{a}$. So $\frac{dE}{dt} = \frac{dU}{dt} = \frac{dU}{da}\frac{da}{dt} = \frac{M\sigma^2}{a}\frac{da}{dt}$.
Using the result of A2 we have

$$\frac{dE}{dt} = -F_{Df}v = -4\pi G^2M^2\frac{\rho(r)}{v}\log\Lambda = -\frac{GM^2\sigma\log\Lambda}{a^2},$$

and the answer is

$$\frac{da}{dt} = -\frac{GM\log\Lambda}{a\sigma} \tag{11}$$

D3. To estimate one can assume that SBHs form a binary when the mass of stars inside the sphere of radius a equals to M:

$$m(a) = \frac{\sigma^2 a}{G} = M,$$

so $a_1 = \frac{GM}{\sigma^2} = 11$ pc.

D4. Integrating the equation (11) we have

$$\frac{a_0^2 - a_1^2}{2} = \frac{GM \log \Lambda}{\sigma} T_1$$

and using that $a_1 \ll a_0$ we have

$$T_1 = \frac{a_0^2 \sigma}{2GM \ln \Lambda} = 0.12\,\mathrm{Gy}.$$

D5. Total energy losses are caused by gravitational slingshot and gravitational waves emission, so

$$\frac{dE}{dt} = \frac{G^2 M^2 \rho}{4\sigma} - \frac{64 G^4 M^5}{5 c^5 a^5} \tag{12}$$

Energy losses due to GW dominates when $\frac{G^2 M^2 \rho_1}{4\sigma} < \frac{64 G^4 M^5}{5 c^5 a^5}$ i.e. $a < a_2$ where

$$a_2^5 = \frac{256}{5} \frac{G^2 M^3 \sigma}{c^5 \rho_1}.$$

Numerical answers are $\rho_1 = 6 \times 10^3 M_S/\mathrm{pc}^3$ and $a_2 = 0.026$ pc.

D6. For rough approximation it can be considered that at the slingshot stage losses are due to slingshot only, so T_2 is calculated analogiously to B5:

$$T_2 = \frac{\sigma}{G \rho a_2} = 0.27\,\mathrm{Gy}$$

And at the GW emission stage losses are due to GW only, so T_3 is calculated directly from to C2:

$$T_3 = \frac{a_2^4 c^5}{16 G^3 M^3} = \frac{\sigma}{16 G \rho a_2} = 0.067\,\mathrm{Gy}$$

D7. Total time of SBH binary evolution from the moment of galaxies merging to SBH merging equals

$$T_{evol} = T_1 + T_2 + T_{GW} = 0.12 + 0.27 + 0.07\,\mathrm{Gy} = 0.46\,\mathrm{Gy}$$