Zero-length springs and slinky coils

A zero effective length spring (ZLS) is a spring for which the force is proportional to the spring's length, $F = kL$ for $L > L_0$ where L_0 is the minimal length of the spring as well as its unstretched length. Figure 1 shows the relation between the force F and the spring length L for a ZLS, where the slope of the line is the spring constant k.

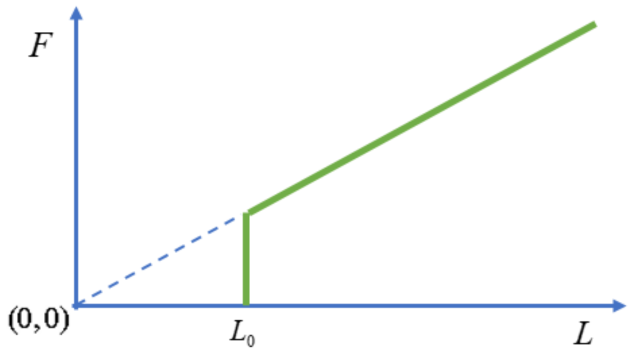

Figure 1: the relation between the force F and the spring length L

A ZLS is useful in seismography and allows very accurate measurement of changes in the gravitational acceleration g. Here, we shall consider a homogenous ZLS, whose weight Mg exceeds kL_0. We define a corresponding dimensionless ratio, $\alpha = kL_0/Mg < 1$ to characterize the relative softness of the spring. The toy known as “slinky” may be (but not necessarily) such a ZLS.

Part A: Statics (3.0 points)

A.1	Consider a segment of length $\Delta\ell$ of the unstretched ZLS spring which is then stretched by a force F, under weightless conditions. What is the length Δy of this segment as a function of $F, \Delta\ell$ and the parameters of the spring?	0.5pt

A.2	For a segment of length $\Delta\ell$, calculate the work ΔW required to stretch it from its original length $\Delta\ell$ to a length Δy.	0.5pt

Throughout this question, we will denote a point on the spring by its distance $0 \leq \ell \leq L_0$ from the bottom of the spring when it is unstretched. In particular, for every point on the spring, ℓ remains unchanged as the spring stretches.

A.3	Suppose that we hang the spring by its top end, so that it stretches under its own weight. What is the total length H of the suspended spring in equilibrium? Express your answers in terms of L_0 and α.	2.0pt

Part B: Dynamics (5.5 points)

Experiments show that when the spring is hung at rest and then released, it gradually contracts from the top, while the lower part remains stationary (see Figure 2). As time advances, the contracting part moves as a solid chunk and accumulates additional turns of the spring, while the stationary part becomes shorter. Every point on the spring begins to move only when the moving part reaches it. The bottom end of the spring starts moving only when the spring is fully collapsed and reaches its unstretched length L_0. After that, the contracted spring continues falling straight downwards, without tumbling, as a rigid body under the influence of gravity.

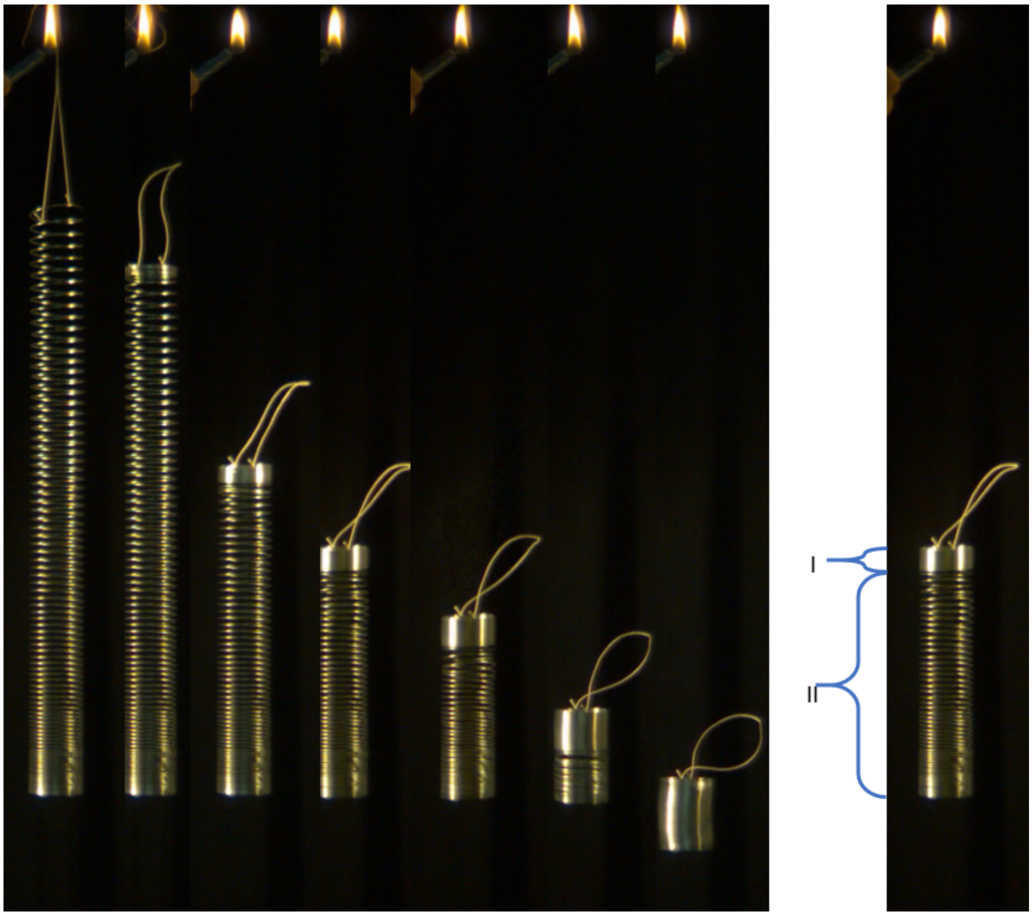

Figure 2: Left: a sequence of pictures taken during the free fall of slinky. Right: the moving part I and the stationary part II during the free fall of the spring.

In the remaining parts of the question, you are asked to base your solution on this described model. You may neglect air resistance, but you are not allowed to neglect L_0.

B.1	Calculate the time t_c it takes from the moment the spring is released, until it fully collapses back to its minimal length L_0. Express your answer in terms of L_0, g and α. Compute the numerical value of t_c for a spring with $k = 1.02\ \mathrm{N/m}$, $L_0 = 0.055\,\mathrm{m}$ and $M = 0.201\,\mathrm{kg}$, while taking g to be $9.80\,\mathrm{m/s^2}$.	2.5pt

B.2	In this task ℓ is used to denote the coordinate of the boundary between parts I (in figure 2, the moving part) and II (the stationary part). At a certain moment, while a stationary part still exists its mass is $m(\ell) = \frac{\ell}{L_0} M$, and the moving part moves with uniform instantaneous velocity $v_I(\ell)$. Show that at this moment (while there exists a stationary part) the velocity of the moving part is $v_I(\ell) = \sqrt{A\ell + B}$. Express the constants A and B in terms of L_0, g and α.	2.5pt

B.3	Based on B.2, find the minimum speed v_{min} of the moving part of the spring in the course of its motion, after its release and before it hits the ground. Express your answer in terms of L_0, α, A and B.	0.5pt

Part C: Energetics (1.5 points)

C.1	Calculate the amount of mechanical energy Q that was lost by generating heat, from the moment the spring is released until just before the spring hits the ground. Express your answer in terms of L_0, M, g and α.	1.5pt