The Physics of a Microwave Oven

This question discusses the generation of microwave radiation in a microwave oven, and its use to heat up food. The microwave radiation is generated in a device called "magnetron". Part A concerns the operation of the magnetron, while part B deals with the absorption of microwave radiation in food.

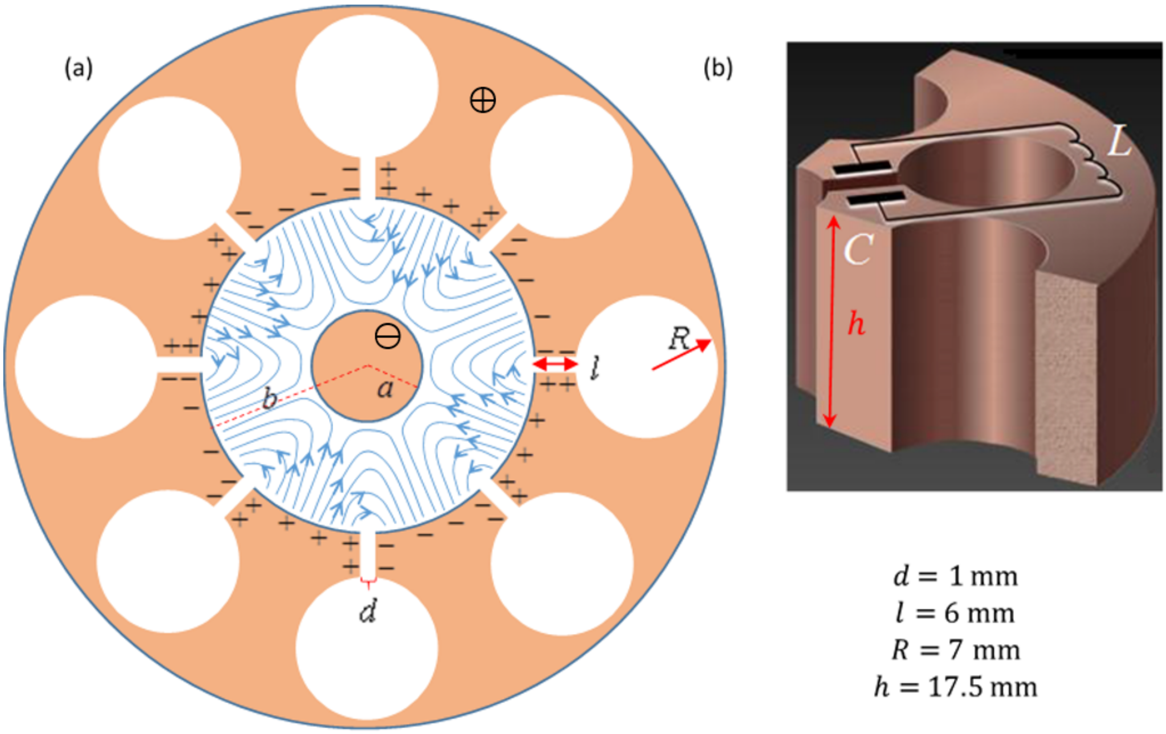

Figure 1

Part A: The structure and operation of a magnetron (6.6 points)

A magnetron is a device for the generation of microwave radiation, either in pulses (for radar applications), or continuously (e.g., in a microwave oven). The magnetron has a mode of self-amplifying oscillations. Supplying the magnetron with static (non-alternating) voltage quickly excites this mode. The microwave radiation thus created is transmitted out of the magnetron.

A typical microwave oven magnetron consists of a solid copper cylindrical cathode (with radius a) and a surrounding anode (with radius b). The latter has the shape of a thick cylindrical shell into which cylindrical cavities are drilled. These cavities are known as "resonators". One of the resonators is coupled to an antenna which will transmit the microwave energy out; we will ignore the antenna in the following. All internal spaces are in vacuum. We will consider a typical magnetron with eight resonators, as depicted in Figure 1(a). The three-dimensional structure of a single resonator is shown in Figure 1(b). As indicated there, each of the eight cavities behaves as an inductor-capacitor (LC) resonator, with operating frequency $f = 2.45\ \mathrm{GHz}$.

A static uniform magnetic field is applied along the magnetron's longitudinal axis, pointing out of the page in Figure 1(a). In addition, a constant voltage is applied between the anode (positive potential) and the cathode (negative potential). Electrons emitted from the cathode reach the anode and charge it, such that they excite an oscillation mode in which the sign of the charge is opposite between every two adjacent resonators. The oscillation of the cavities amplify these oscillations.

The process described above creates an alternating electric field with the aforementioned frequency $f = 2.45\ \mathrm{GHz}$ (blue lines in Figure 1(a); the static field is not plotted) in the space between the cathode and the anode, in addition to the static field caused by the applied constant voltage. In the steady state, the typical amplitude of the alternating electric field between the anode and the cathode is approximately $\frac{1}{3}$ of the static electric field there. The electron motion in the space between the cathode and the anode is affected by both the static and the alternating parts of the field. This causes electrons that reach the anode to transfer about 80% of the energy they acquire from the static field into the alternating field. A minority of the ejected electrons returns to the cathode and releases additional electrons, further amplifying the alternating field.

Each resonator can be thought of as a capacitor and an inductor, see Figure 1(b). The capacitance mainly arises from the planar parts of the resonator surface, while the inductance stems from the cylindrical part. Assume that the current in the resonator flows uniformly very close to the surface of its cylindrical cavity, and that the strength of the magnetic field generated by this current is 0.6 times that of an ideal infinite solenoid. The various lengths defining the resonator geometry are given in Figure 1(b). The vacuum permittivity and permeability are $\varepsilon_0 = 8.85 \cdot 10^{-12}\,\frac{\mathrm{F}}{\mathrm{m}}$ and $\mu_0 = 4\pi \cdot 10^{-7}\,\frac{\mathrm{H}}{\mathrm{m}}$, respectively.

A.1	Use the above data to estimate the frequency f_{est} of a single resonator. (Your result may differ from the actual value, $f = 2.45\ \mathrm{GHz}$. Use the **actual** value in the remainder of the question.)	0.4pt

Task A.2 below does not deal with the magnetron itself, but helps to introduce some of the relevant physics. Consider an electron moving in free space under the influence of a uniform electric field directed along the negative y axis, $\vec{E} = -E_0\hat{y}$, and a uniform magnetic field directed along the positive z axis, $\vec{B} = B_0\hat{z}$ (E_0 and B_0 are positive; $\hat{x}, \hat{y}, \hat{z}$ are unit vectors oriented in the conventional manner). Let us denote the electron velocity at time t by $\vec{u}(t)$. The drift velocity $\vec{u}_D$ of the electron is defined as its average velocity. We denote by m and $-e$ the mass and charge of the electron, respectively.

A.2	In each of the following two cases, find $\vec{u}_D$. In addition, draw in the Answer Sheet the electron's trajectory (in the lab frame) during the time interval $0 < t < \frac{4\pi m}{eB_0}$ if: 1. at $t = 0$ the electron velocity is $\vec{u}(0) = (3E_0/B_0)\hat{x}$, 2. at $t = 0$ the electron velocity is $\vec{u}(0) = -(3E_0/B_0)\hat{x}$.	1.5pt

We now resume our discussion of the magnetron. The distance between the cathode and the anode is $15\,\mathrm{mm}$. Assume that, due to the aforementioned energy loss to the alternating fields, the maximal kinetic energy of each electron does not exceed $K_{\max} = 800\,\mathrm{eV}$. The static magnetic field strength is $B_0 = 0.3\,\mathrm{T}$. The electron mass and charge are $m = 9.1 \cdot 10^{-31}\,\mathrm{kg}$ and $-e = -1.6 \cdot 10^{-19}\,\mathrm{C}$, respectively.

A.3	Numerically estimate the maximal radius r of the electron motion trajectory in the reference frame in which this motion is approximately circular, considering this reference frame as approximately inertial.	0.4pt

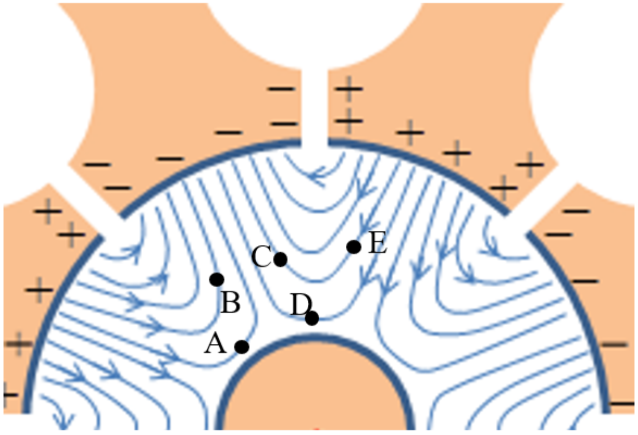

Figure 2

A.4	Figure 2 depicts the alternating electric field lines between the anode and the cathode at a given moment in time (the static field is not plotted). Indicate in the Answer Sheet which of the electrons positioned at A,B,C,D and E will drift towards the anode, which will drift towards the cathode and which will drift at a direction perpendicular to the radius at that moment.	1.2pt

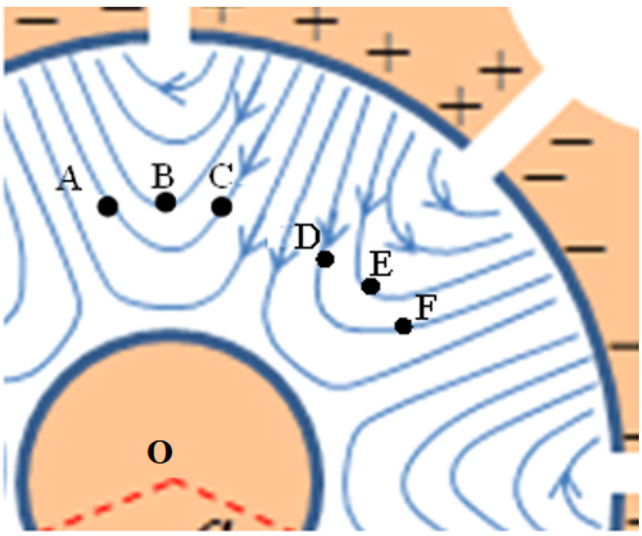

Figure 3

Figure 3 depicts the alternating electric field lines between the anode and the cathode (the static field is not plotted) at a given moment in time. The positions of six electrons at that moment are denoted by A, B, C, D, E and F. All electrons are at the same distance from the cathode.

A.5	Consider the situation shown in Figure 3. For each of the six electron pairs AB, AC, BC, DE, DF, EF, indicate in the Answer Sheet whether their drift will cause the angle between their position vectors (measured from the cathode's center O) to increase or decrease at that moment.	1.2pt

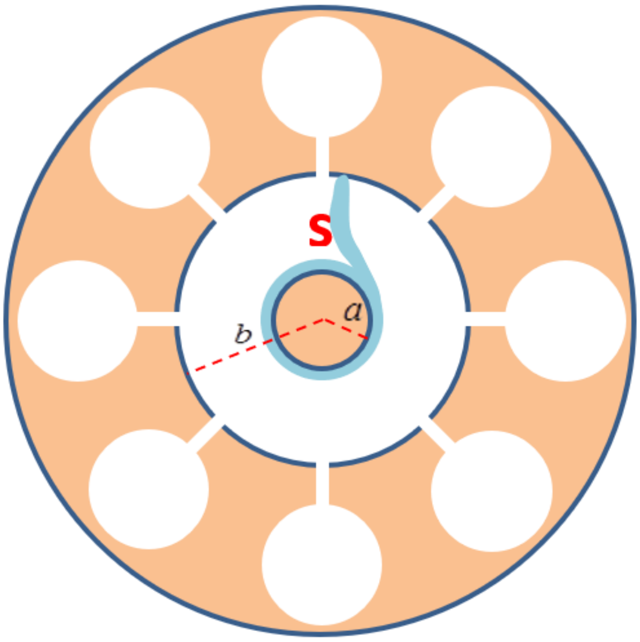

Figure 4

The pattern you have discovered in Task A.5 acts as a focusing mechanism, concentrating the electrons in the space between the cathode and anode into spokes. Figure 4 depicts one such spoke, denoted by S.

A.6	Depict in the Answer Sheet the other spokes at that moment. Indicate by arrows their direction of rotation, and calculate their average angular velocity ω_s.	0.8pt

Make the approximation that the total electric field half-way between the cathode and the anode is equal to its average static value along a radial line from the cathode to the anode, and that the spokes are approximately radial in that region. The cathode and anode radii (a and b, respectively) are defined in Figure 4.

A.7	Find an approximate expression for the static voltage V_0 required for operating the magnetron in the manner described. (The expression you will find gives an approximation for the minimal value required for the magnetron operation; the optimal voltage is somewhat higher.)	1.1pt

Part B: The interaction of microwave radiation with water molecules (3.4 points)

This part deals with the usage of microwave radiation (radiated by the magnetron antenna into the food chamber) for cooking, that is, heating up a lossy dielectric material such as water, either pure or salty (which is our model for, say, soup).

An electric dipole is a configuration of two equal and opposite electric charges q and $-q$ a small distance d apart. The electric dipole vector points from the negative to the positive charge, and its magnitude is $p = qd$.

A time-dependent electric field $\vec{E}(t) = E(t)\hat{x}$ is applied on a single dipole of moment $\vec{p}(t)$ with constant magnitude $p_0 = |\vec{p}(t)|$. The angle between the dipole and the electric field is $\theta(t)$.

B.1	Write expressions for both the magnitude of the torque $\tau(t)$ applied by the electric field on the dipole and the power $H_i(t)$ delivered by the field to the dipole, in terms of p_0, $E(t)$, $\theta(t)$ and their derivatives.	0.5pt

Water molecules are polar, hence can be treated as electric dipoles. Due to the strong hydrogen bonds between water molecules in liquid water, one cannot treat them as independent dipoles. Rather, one should refer to the polarization vector $\vec{P}(t)$, which is the dipole moment density (average dipole moment per unit volume of an ensemble of water molecules). The polarization $\vec{P}(t)$ is parallel to the local applied alternating electric field (of the microwave radiation), $\vec{E}(t)$, and oscillates in time with an amplitude that is proportional to the amplitude of the local alternating electric field, but with a phase lag δ.

The local alternating electric field at a given location inside the water is $\vec{E}(t) = E_0 \sin(\omega t)\hat{x}$, where $\omega = 2\pi f$, giving rise to polarization $\vec{P}(t) = \beta\varepsilon_0 E_0 \sin(\omega t - \delta)\hat{x}$, where the dimensionless constant β is a property of water.

B.2 Find an expression for the time-averaged power $\langle H(t)\rangle$ per unit volume absorbed by the water. **0.5pt**

The time-average for a time dependent periodic variable $f(t)$ over its period T is defined as:

$$\langle f(t)\rangle = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,\mathrm{d}t. \tag{1}$$

Let us now consider the propagation of the radiation through the water. The relative dielectric constant of water (at the electromagnetic field frequency) is ε_r, and the corresponding index of refraction of water is $n = \sqrt{\varepsilon_r}$. The momentary energy density of the electric field is given by $\frac{1}{2}\varepsilon_r\varepsilon_0 E^2$. The time-averaged energy density of the electric and magnetic fields are equal.

B.3	Let us denote the time-averaged radiation energy flux density by $I(z)$ (average radiation power flow per unit area). Here z is the depth of penetration into the water, and the radiation propagates in the z direction. Find an expression for the dependence of the flux density $I(z)$ on z. The flux density at the water surface, $I(0)$, may appear in your result.	1.1pt

The phase lag δ is the result of the interaction between the water molecules. It depends on the dimensionless dielectric loss coefficient ε_ℓ and the relative dielectric constant ε_r (both of which depend on the radiation angular frequency ω and the temperature) via the relation $\tan\delta = \varepsilon_\ell/\varepsilon_r$. When δ is small enough, the electric field at penetration depth z into the water is given by:

$$\vec{E}(z,t) = \vec{E}_0 e^{-\frac{1}{2}nk_0 z\tan\delta}\sin(nk_0 z - \omega t) \tag{2}$$

where $k_0 = \omega/c$ and $c = 3.0\cdot 10^8\,\frac{m}{s}$ is the speed of light in vacuum.

B.4	Employ the approximation $\tan\delta \approx \sin\delta$ and find an expression for the coefficient β defined in Task B.2 in terms of the other parameters.	0.6pt

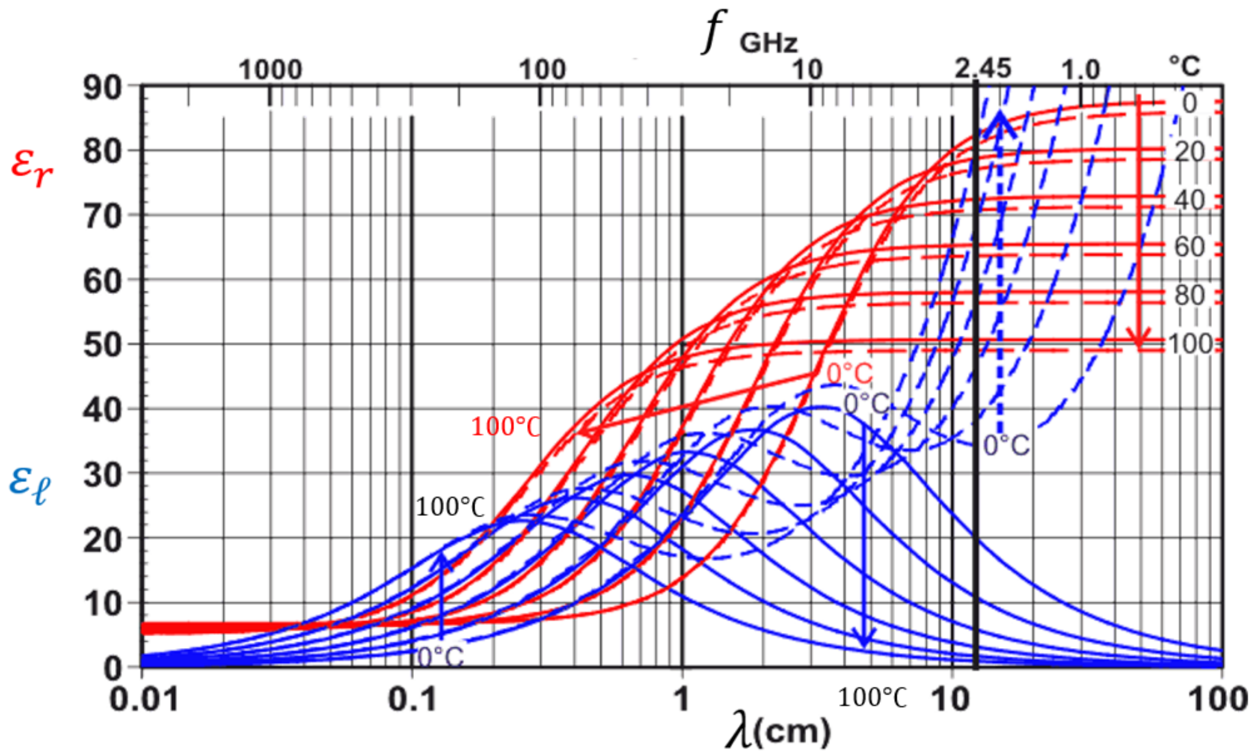

Figure 5. The arrows indicate the variation with temperature across the curves from $0^\circ\mathrm{C}$ to $100^\circ\mathrm{C}$.

Figure 5 depicts ε_ℓ (blue) and ε_r (red) for both pure water (solid lines) and a dilute solution of salt in water (dashed lines) as functions of wavelength or frequency, at several different temperatures. The angular frequency $\omega = 2\pi \cdot 2.45 \cdot 10^9\ \mathrm{s}^{-1}$ is indicated by a bold vertical line. Below we will consider microwave radiation at this frequency only.

B.5 Use Figure 5 to address the following questions: (0.7pt)

1. For water at $20^\circ\mathrm{C}$, find the penetration depth $z_{1/2}$ at which the power per unit volume is reduced to half of its value at $z = 0$.
2. Indicate in the Answer Sheet whether the penetration depth of the microwave radiation into water increases, decreases or remains the same with temperature.
3. Indicate in the Answer Sheet whether the penetration depth of the microwave radiation into soup (dilute salt solution) increases, decreases or remains the same with temperature.