Theory Q1, APhO 2019, Adelaide
RF Reflectometry of Single-Electron Circuits

Version 1.0.

PART A: LUMPED ELEMENT MODEL OF A CO-AXIAL TRANSMISSION LINE

A.1 The speed of wave propagation in free spaces ($c_0 = 299\,792\,458\,\mathrm{m/s}$) is $c_0 = 1/\sqrt{\varepsilon_0\,\mu_0}$. The speed in the dielectric & diamagnetic medium is

$$v = \frac{1}{\sqrt{\varepsilon_\mathrm{r}\,\varepsilon_0\,\mu_\mathrm{r}\,\mu_0}} = \frac{c_0}{\sqrt{\varepsilon_\mathrm{r}\,\mu_\mathrm{r}}} \tag{1}$$

A.2 Gauss law for the flux through a cylindrical surface with radius r co-axial with the the core, $a < r < b$:

$$\Delta l\, 2\pi r\, E(r) = \frac{\Delta q}{\varepsilon_\mathrm{r}\varepsilon_0} \Rightarrow E(r) = \frac{\Delta q}{\Delta l}\frac{1}{2\pi\varepsilon_\mathrm{r}\varepsilon_0 r} \tag{2}$$

A.3 The capacitance

$$c\,\Delta l = \frac{\Delta q}{\phi} \tag{3}$$

where the potential ϕ of the core with respect to the shield is

$$0 - \phi = -\int_a^b E(r)\,dr \Rightarrow \phi = \frac{\Delta q}{\Delta l}\frac{1}{2\pi\varepsilon_\mathrm{r}\varepsilon_0}\ln\frac{b}{a} \tag{4}$$

$$c = \frac{2\pi\varepsilon_\mathrm{r}\varepsilon_0}{\ln\frac{b}{a}} \tag{5}$$

A.4 The magnetic flux through a rectangular contour paralel to the axis equal inductance times the current:

$$\Delta l \int_a^b B(r)\,dr = l\,\Delta l\, I \tag{6}$$

Biot-Savart law $B(r) = \frac{\mu_\mathrm{r}\mu_0}{2\pi}\frac{I}{r}$ gives

$$l = \frac{\mu_\mathrm{r}\mu_0}{2\pi}\ln\frac{b}{a} \tag{7}$$

A.5 i. Adding Δl length of the cable should not change its impedance. Hence the impedance Z of the following circuit must be equal to Z_0:

$$\frac{1}{Z} = \frac{1}{Z_0 + j\omega\Delta L} + \frac{1}{\frac{1}{j\omega\Delta C}} = \frac{1}{Z_0} \tag{8}$$

$$Z_0 + i\omega\Delta L\, Z_0 - \Delta L/\Delta C = 0 \tag{9}$$

$\Delta L/\Delta C = l/c$ and the $\Delta L \to 0$ for $\Delta l \to 0$, hence

$$Z_0 = \sqrt{l/c} \tag{10}$$

ii.

$$Z_0 = \sqrt{l/c} = \frac{\ln(b/a)}{2\pi}\sqrt{\frac{\mu_\mathrm{r}\mu_0}{\varepsilon_\mathrm{r}\varepsilon_0}} = \ln(b/a)\sqrt{\frac{\mu_\mathrm{r}}{\varepsilon_\mathrm{r}}} \times 59.96\,\Omega \tag{11}$$

For $Z_0 = 50\,\Omega$, $\varepsilon_\mathrm{r} = 4.0$ and $\mu_\mathrm{r} = 1.0$ this gives $b = 5.30\,a$.

PART B: HYPOTHETICAL TRANSMISSION LINE WITH RETURN ALONG A GROUNDED PLANE

B.1 The high-conductance ground plate can be replaced by an image of the wire with opposite direction of the current at distance $2d$ from the real wire. The magnetic fields from the real and the imaginary wires add up and need to be integrated to get the magnetic flux between the wire and the plate:

$$l\,\Delta l\,I = \frac{\mu\mu_0}{2\pi} I \int_a^d \underbrace{\left(\frac{1}{r} + \frac{1}{2d-r}\right)}_{\approx 2d/r} dr\,\Delta l \tag{12}$$

$$l = \frac{\mu\mu_0}{2\pi} \ln\frac{2d}{a} \tag{13}$$

The potential difference between the wire and the plate can be obtained similarly by integrating the combined field for the wire and its image:

$$\phi = \frac{\Delta q}{\Delta l}\frac{1}{2\pi\varepsilon_\mathrm{r}\varepsilon_0}\int_a^d \left(\frac{1}{r} + \frac{1}{2d-r}\right) dr = \frac{\Delta q}{\Delta l}\frac{\ln(2d/a)}{2\pi\varepsilon_\mathrm{r}\varepsilon_0} \tag{14}$$

$$c = \frac{\Delta q}{\Delta l}\frac{1}{\phi} = \frac{2\pi\varepsilon_\mathrm{r}\varepsilon_0}{\ln(2d/a)} \tag{15}$$

Hence the impedance of the wire-plate system is

$$Z_0 = \sqrt{l/c} = \frac{\ln(2d/a)}{2\pi}\sqrt{\frac{\mu_\mathrm{r}\mu_0}{\varepsilon_\mathrm{r}\varepsilon_0}} = \ln(2d/a)\sqrt{\frac{\mu_\mathrm{r}}{\varepsilon_\mathrm{r}}} \times 59.96\,\Omega \tag{16}$$

PART C: BASICS OF RF REFLECTOMETRY

C.1 At the interface, values of the voltage on both transmission lines have to coincide:

$$V_\mathrm{i} + V_\mathrm{r} = V_\mathrm{r} \tag{17}$$

The current has to be conserved at the interface, however, the incident and the reflected waves carry the current in opposite directions:

$$\frac{V_\mathrm{i}}{Z_0} - \frac{V_\mathrm{r}}{Z_0} = \frac{V_\mathrm{t}}{Z_1} \tag{18}$$

It is clear from the equation above that $V_\mathrm{t} \neq 0$ if $Z_0 \neq Z_1$ – impedance mismatch has to cause reflection. Solving the voltage and the current equations for $\Gamma = V_\mathrm{r}/V_\mathrm{i}$ gives

$$\Gamma = \frac{Z_1 - Z_0}{Z_1 + Z_0} \tag{19}$$

C.2 A π-shift implies opposite signs of V_i and V_r and hence requires $\Gamma < 0$. This implies $Z_1 < Z_0$.

PART D: THE SINGLE ELECTRON TRANSISTOR

D.1 i. The potential on the QD is

$$\varphi_n = V_g + \frac{-ne}{C_\mathrm{g}} \tag{20}$$

where $e > 0$ is the elementary charge.

ii. Bringing the additional electron decreases the potential to $\varphi_{n+1} = \varphi_n - e/C_g$. The work necessary to perform the transition is

$$\Delta E_n = -e\frac{\varphi_n + \varphi_{n+1}}{2} = \frac{e^2}{C_g}\left(n + \frac{1}{2}\right) - eV_g \tag{21}$$

Note that without $C_t \ll C_g$ approximation, the answer is $\Delta E_n = \frac{e^2}{C_g+C_t}\left(n+\frac{1}{2}\right) - eV_g C_g/(C_t + C_g)$.

D.2 Consider such V_g that $\Delta E_{\mathcal{N}} \to 0^+$. Then $\Delta E_{\mathcal{N}+1} = \frac{e^2}{C_g}$ can become the new equilibrium addition energy once V_g is made slightly more positive another electron jumps onto the dot, $\mathcal{N} \to \mathcal{N}+1$. Further increase of V_g only lowers the addition energy till the next resonance condition is reached. In other words, $\Delta E_{\mathcal{N}(V_g)}(V_g)$ is a periodic function of V_g with a discontinuity at the maximal value of

$$E_c = \frac{e^2}{C_g} \tag{22}$$

D.3 Thermal energy per electron $k_{\mathrm{B}}T$ should not exceed characteristic addition energy E_c:

$$k_{\mathrm{B}}T < E_c \tag{23}$$

Here k_{B} is the Boltzmann constant.

D.4 i. $\tau = R_t\, C_t$

ii. Quantum uncertainty of energy (life-time broadening) h/τ must be less the energy difference between the states with n and $n+1$ electrons,

$$h/\tau < E_c \tag{24}$$

$$\frac{h}{R_t C_t} < \frac{e^2}{C_t + C_g} \tag{25}$$

$$R_t > \frac{h}{e^2}\frac{C_t + C_g}{C_t} > \frac{h}{e^2} \tag{26}$$

PART E. RF REFLECTOMETRY TO READ OUT SET STATE

E.1

$$\Gamma = \frac{Z_{\mathrm{SET}} - Z_0}{Z_{\mathrm{SET}} + Z_0} \tag{27}$$

$$\Gamma_{\mathrm{ON}} = \frac{10^5 - 50}{10^5 + 50} \approx 1 - 2\frac{50}{10^5} \tag{28}$$

$$\Gamma_{\mathrm{OFF}} = \lim_{Z_1\to\infty}\frac{Z_1 - Z_0}{Z_1 + Z_0} = 1 \tag{29}$$

$$\Delta\Gamma = |\Gamma_{\mathrm{ON}} - \Gamma_{\mathrm{OFF}}| \approx 1.0 \cdot 10^{-3} \tag{30}$$

E.2 In the OFF state of the SET, the circuit is an LC contour with resonance frequency $\omega_0 = 1/\sqrt{L_0 C_0}$. If we choose L_0 such that $\omega_0 = \omega_{\mathrm{rf}}$, then Z_{tot} (the total impedance of the circuit) in the OFF state of the SET equals to 0 and the reflectance is that of a short-circuited line, $\Gamma_{\mathrm{OFF}} = -1$.

The change in reflectance will be large if $|Z_{\mathrm{tot}}|$ is the ON state is on the order of Z_0 or larger, which is indeed the case. For the ON state and $\omega_0 = \omega_{\mathrm{rf}}$

$$Z_{\mathrm{tot}} = \left(\frac{1}{\frac{1}{j\omega C}} + \frac{1}{R_{\mathrm{SET}}}\right)^{-1} + j\omega L \tag{31}$$

$$= \frac{R_{\mathrm{SET}}}{1 + j\omega C\, R_{\mathrm{SET}}} + j\omega L \tag{32}$$

$$= \frac{R_{\mathrm{SET}} + i\sqrt{L/C}}{1 + R_{\mathrm{SET}}^2 C/L} \tag{33}$$

For $C_0 = 0.4 \cdot 10^{-12}$F, $Z_0 = 50\,\Omega$ and $\omega_{\rm rf} = 2\pi \cdot 10^8$ Hz, we have $L_0 = 6.33$ mH, $Z_{\rm tot} = (158 + 6.3\,j)\Omega$, $\Gamma_{\rm ON} = 0.5198 + 0.0145\,j$, and $\Delta\Gamma = 1.52$.

PAR F. CHARGE SENSING WITH A SINGLE LEAD QUANTUM DOT

F.1 The SLQD readout circuit contains only reactive elements, so $|\Gamma| = 1$ will always be zero. The OFF state of the SLQD corresponds to an inductor L_0 and a capacitor C_0 connected in series. We again choose $\omega_{\rm rf} = 1/\sqrt{L_0 C_0}$, so that $Z_{\rm tot}$ is the OFF state is infinite and $\Gamma_{\rm OFF} = 1$.

The ON state corresponds to $Z_{\rm SET} = -j\frac{1}{\omega_{\rm rf} C_q}$ and $Z_{\rm tot}$ at $\omega_{\rm rf} = \omega_0$ is just the impedance of the SLQD

$$Z_{\rm tot} = \frac{1}{(j\omega_{\rm rf} L_0)^{-1} + j\omega_{\rm rf}(C + C_q)} = -j\frac{1}{\omega_0 C_q} = -j\frac{C_0}{C_q} Z_C \tag{34}$$

For the phase of $\Gamma_{\rm ON} = (Z_{\rm tot} - Z_0)/(Z_{\rm tot} + Z_0)$ to be significantly different from zero, we need $|Z_{\rm tot}| \sim Z_0$ since $Z_{\rm tot}$ is purely imaginary. Hence

$$Z_C \sim \frac{C_q}{C_0} Z_0 \tag{35}$$

F.2 If L_0 is fixed, we can still operate the circuit at the frequency $\omega_{\rm rf} = 1/\sqrt{L_0 C_0}$ that gives $\Gamma_{\rm OFF} = 1$. However, we need to deduce a way to increase $|Z_{\rm tot}|$ even if Z_C is not sufficient. One of the ways to do that is to add an additional capacitance C_m is series with rest of the circuit.

This will give (at $\omega_{\rm rf} = \omega_0$)

$$Z_{\rm tot} = -j\left(\frac{C_0}{C_q} Z_C + \frac{1}{\omega_0 C_m}\right) = -j\omega_0^{-1}\left(C_q^{-1} + C_m^{-1}\right) \tag{36}$$

From the condition $|Z_{\rm tot}| \approx Z_0$ we have

$$C_m \approx \frac{C_q}{Z_0 C_q \omega_0 - 1} = \frac{C_q\sqrt{L_0 C_0}}{Z_0 C_q - \sqrt{L_0 C_0}} \tag{37}$$