Q3 SOLUTIONS: TIPPE TOP

1. (a) **Reference sheet for markers**

Coordinate systems for convenience (note: use of matrices not needed) xyz from XYZ

$$\begin{bmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{y}}\\ \hat{\mathbf{z}}\end{bmatrix} = \begin{bmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{\mathbf{X}}\\ \hat{\mathbf{Y}}\\ \hat{\mathbf{Z}}\end{bmatrix}$$

123 from xyz

$$\begin{bmatrix}\hat{\mathbf{1}}\\ \hat{\mathbf{2}}\\ \hat{\mathbf{3}}\end{bmatrix} = \begin{bmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{bmatrix}\begin{bmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{y}}\\ \hat{\mathbf{z}}\end{bmatrix}$$

Position of point A from centre of mass, in xyz and 123 frames:

$$\begin{aligned}\mathbf{a} &= \alpha R\hat{\mathbf{3}} - R\hat{\mathbf{z}} \qquad (1)\\ &= \alpha R\sin\theta\hat{\mathbf{x}} + R(\alpha\cos\theta - 1)\hat{\mathbf{z}}\\ &= R\sin\theta\hat{\mathbf{1}} + R(\alpha - \cos\theta)\hat{\mathbf{3}}\end{aligned}$$

Useful products:

$$\begin{aligned}\hat{\mathbf{z}}\times\hat{\mathbf{3}} &= \sin\theta\hat{\mathbf{y}} \qquad (2)\\ \hat{\mathbf{z}}\cdot\hat{\mathbf{3}} &= \cos\theta \qquad (3)\\ \hat{\mathbf{x}}\cdot\hat{\mathbf{3}} &= \sin\theta \qquad (4)\\ \mathbf{a}\cdot\hat{\mathbf{3}} &= R(\alpha - \cos\theta) \qquad (5)\\ \mathbf{a}\cdot\hat{\mathbf{z}} &= R(\alpha\cos\theta - 1) \qquad (6)\end{aligned}$$

Note (given in question):

$$\left(\frac{\partial\mathbf{A}}{\partial t}\right)_{\mathbf{K}} = \left(\frac{\partial\mathbf{A}}{\partial t}\right)_{\widetilde{\mathbf{K}}} + \boldsymbol{\omega}\times\mathbf{A} \qquad (7)$$

Time derivatives:

$$\begin{aligned}\dot{\hat{\mathbf{3}}} &= \boldsymbol{\omega}\times\hat{\mathbf{3}} \qquad (8)\\ \dot{\hat{\mathbf{x}}} &= \dot{\phi}\hat{\mathbf{y}} \qquad (9)\\ \dot{\hat{\mathbf{y}}} &= -\dot{\phi}\hat{\mathbf{x}} \qquad (10)\\ & \qquad (11)\end{aligned}$$

Triple product (given in question):

$$\begin{aligned}\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) &= \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})\\ (\mathbf{A}\times\mathbf{B})\times\mathbf{C} &= -\mathbf{A}(\mathbf{B}\cdot\mathbf{C}) + \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) \qquad (12)\end{aligned}$$

(b) **Solutions**

i. (**0.8 marks**)

For free body diagram, refer to figure at end

$$\begin{aligned}\sum \mathbf{F}_{\text{ext}} &= (N - mg)\hat{z} + \mathbf{F}_f \quad \text{(full marks)} \qquad (13)\\ &= (N - mg)\hat{z} - \frac{\mu_k N}{|v_A|}\mathbf{v_A}\end{aligned}$$

Sketched $\mathbf{v_A}$ must be in opposite direction to $\mathbf{F}_f$.

ii. (**0.8 marks**)

$$\begin{aligned}\sum \boldsymbol{\tau}_{\text{ext}} &= \mathbf{a} \times (N\hat{\mathbf{z}} + \mathbf{F}_f) \qquad (14)\\ &= (\alpha R\hat{\mathbf{3}} - R\hat{\mathbf{z}}) \times (N\hat{\mathbf{z}} + F_{f,x}\hat{\mathbf{x}} + F_{f,y}\hat{\mathbf{y}})\\ &= \alpha RN\hat{\mathbf{3}} \times \hat{\mathbf{z}} + \alpha R(\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{z}}) \times (F_{f,x}\hat{\mathbf{x}} + F_{f,y}\hat{\mathbf{y}}) - R\hat{\mathbf{z}} \times (F_{f,x}\hat{\mathbf{x}} + F_{f,y}\hat{\mathbf{y}})\\ &= -\alpha RN\sin\theta + \alpha R\sin\theta F_{f,y}\hat{\mathbf{z}} + \alpha R\cos\theta F_{f,x}\hat{\mathbf{y}} - \alpha R\cos\theta F_{f,y}\hat{\mathbf{x}} - RF_{f,x}\hat{\mathbf{y}} + RF_{f,x}\hat{\mathbf{x}}\\ &= RF_{f,y}(1 - \alpha\cos\theta)\hat{\mathbf{z}} + \left[RF_{f,x}(\alpha\cos\theta - 1) - \alpha RN\sin\theta\right]\hat{\mathbf{y}} + \alpha R\sin\theta F_{f,y}\hat{\mathbf{z}} \qquad (15)\end{aligned}$$

iii. (**0.4 marks**)

Motion at A satisfies

$$\mathbf{v_A} = \dot{\mathbf{s}} + \boldsymbol{\omega} \times \mathbf{a} \qquad (16)$$

Contact condition:

$$(\mathbf{s} + \mathbf{a}) \cdot \hat{z} = 0 \qquad (17)$$

Differentiating w.r.t. time, using hint (7) (also $\dot{\hat{\mathbf{z}}} = 0$)

$$(\dot{\mathbf{s}} + \boldsymbol{\omega} \times \mathbf{a}) \cdot \hat{\mathbf{z}} = 0 \qquad (18)$$

The LHS is precisely $\mathbf{v_A} \cdot \hat{\mathbf{z}}$.

iv. (**0.7 marks**)

$$\begin{aligned}\boldsymbol{\omega} &= \dot{\theta}\hat{\mathbf{2}} + \dot{\phi}\hat{\mathbf{z}} + \dot{\psi}\hat{\mathbf{3}}\\ &= \dot{\psi}\sin\theta\hat{\mathbf{x}} + \dot{\theta}\hat{\mathbf{y}} + (\dot{\psi}\cos\theta + \dot{\phi})\hat{\mathbf{z}} \qquad (19)\\ &= -\dot{\phi}\sin\theta\hat{\mathbf{1}} + \dot{\theta}\hat{\mathbf{2}} + (\dot{\psi} + \dot{\phi}\cos\theta)\hat{\mathbf{3}} \qquad (20)\end{aligned}$$

v. (**0.8 marks**)

Where $\mathbf{I}$ is the inertia tensor,

$$\begin{aligned}E_{\text{tot}} &= K_{\text{rot}} + K_{\text{trans}} + U_{\text{grav}}\\ &= \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I}\boldsymbol{\omega} + \frac{1}{2}m\dot{\mathbf{s}}^2 + mgR(1 - \alpha\cos\theta)\end{aligned}$$

From 18,

$$\begin{aligned}\dot{\mathbf{s}} &= \mathbf{v_A} - \boldsymbol{\omega} \times \mathbf{a} \\ &= \mathbf{v_A} - (\dot\theta\hat{\mathbf{2}} + \dot\phi\hat{\mathbf{z}} + \dot\psi\hat{\mathbf{3}}) \times (\alpha R\hat{\mathbf{3}} - R\hat{\mathbf{z}}) \\ &= v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} - \left(\dot\theta\alpha R\hat{\mathbf{1}} - \dot\theta R\hat{\mathbf{z}} + \dot\phi\alpha R\hat{\mathbf{z}} \times \hat{\mathbf{3}} - \dot\psi\hat{\mathbf{3}} \times \hat{\mathbf{z}}\right) \\ &= \left(v_x + \dot\theta R(1-\alpha\cos\theta)\right)\hat{\mathbf{x}} + \left(v_y - R\sin\theta(\alpha\dot\phi - \dot\psi)\right)\hat{\mathbf{y}} + \dot\theta\alpha R\sin\theta\hat{\mathbf{z}}\end{aligned}$$

using 2. Thus

$$\begin{aligned}E_{\text{tot}} = &\frac{1}{2}\left[I_1(\dot\phi^2\sin^2\theta + \dot\theta^2) + I_3(\dot\psi + \dot\phi\cos\theta)^2\right] \\ &+ \frac{m}{2}\left[\left(v_x + \dot\theta R(1-\alpha\cos\theta)\right)^2 + \left(v_y - R\sin\theta(\alpha\dot\phi - \dot\psi)\right)^2 + \dot\theta^2\alpha^2R^2\sin^2\theta\right] + mgR(1-\alpha\cos\theta)\end{aligned}$$

vi. (**0.3 marks**)

From 15,

$$\frac{d\mathbf{L}}{dt}\cdot\hat{\mathbf{z}} = \sum \boldsymbol{\tau}\cdot\hat{\mathbf{z}} = \alpha R\sin\theta F_{f,y} \tag{21}$$

vii. (**1.0 marks**)

Centre of mass of top rises, therefore $U_{\text{grav}} \sim 0$ increases.

Normal force does no work. Frictional force does work at point A, and decreases the total energy monotonically:

$$W = \int \mathbf{F}\cdot\mathbf{v_A}\;dt < 0$$

Frictional force points in opposite direction to $\mathbf{v_A}$:

$$\frac{d}{dt}E_{\text{tot}} = -\mu_k N|\mathbf{v_A}|$$

At start and end (phases I and V) there is little translation so $K_{\text{trans}} \sim 0$; the predominant energy transfer is from $K_{\text{rot}} \sim 0$ to $U_{\text{grav}} \sim 0$.

21 implies only the y-component of the frictional force acts to decrease the component of the angular momentum about the z-axis, thereby transferring energy from the K_{rot} term to the U_{grav} term. The resultant torque is in the $\mathbf{a}\times\hat{\mathbf{y}}$ direction.

viii. (**1.2 marks**)

See attached sketches. Required:

- E_{tot}: monotonically decreasing
- K_{rot}: monotonically decreasing; zero at V
- K_{trans}: zero at I and V; higher between; close to zero at IV
- U_{grav}: flat at start and finish; higher at end; increases from I to IV then flat; increase roughly at same time that K_{rot} decreases

ix. (**0.4 marks**)

From 20,

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega} = I_1\left(-\dot\phi\sin\theta\hat{\mathbf{1}} + \dot\theta\hat{\mathbf{2}}\right) + I_3(\dot\psi + \dot\phi\cos\theta)\hat{\mathbf{3}} \tag{22}$$

Taking cross product with $\hat{\mathbf{3}}$:

$$\begin{aligned}\mathbf{L}\times\hat{\mathbf{3}} &= I_1\left(\dot\phi\sin\theta\hat{\mathbf{2}} + \dot\theta\hat{\mathbf{1}}\right) \\ &= I_1(\boldsymbol{\omega}\times\hat{\mathbf{3}})\end{aligned} \tag{23}$$

x. (**2.1 marks**)

Using Newton's third laws with 13 and 15:

$$m\ddot{\mathbf{s}} = (N - mg)\hat{z} + \mathbf{F}_f \quad (24)$$

$$\frac{d\mathbf{L}}{dt} = \mathbf{a} \times (N\hat{\mathbf{z}} + \mathbf{F}_f) \quad (25)$$

Differentiating 18 again:

$$\begin{aligned}
0 &= \left(\ddot{\mathbf{s}} + \frac{d}{dt}(\boldsymbol{\omega} \times \mathbf{a})\right) \cdot \hat{\mathbf{z}} \qquad (26)\\
&= \ddot{\mathbf{s}}.\hat{\mathbf{z}} + \frac{d}{dt}\left(\boldsymbol{\omega} \times \alpha R\hat{\mathbf{3}} - \boldsymbol{\omega} \times R\hat{\mathbf{z}}\right).\hat{\mathbf{z}} \quad \text{using 1}\\
&= \frac{-mg + N}{m} + \frac{\alpha R}{I_1}\frac{d}{dt}(\mathbf{L} \times \hat{\mathbf{3}}).\hat{\mathbf{z}} - R\left(\frac{d\boldsymbol{\omega}}{dt} \times \hat{\mathbf{z}}\right) \cdot \hat{\mathbf{z}} \quad \text{using 24, 23, and } \dot{\hat{\mathbf{z}}} = 0\\
&= \frac{-mg + N}{m} + \frac{\alpha R}{I_1}\left(\frac{d\mathbf{L}}{dt} \times \hat{\mathbf{3}}\right) \cdot \hat{\mathbf{z}} + \frac{\alpha R}{I_1}\left(\mathbf{L} \times \frac{d\hat{\mathbf{3}}}{dt}\right) \cdot \hat{\mathbf{z}}
\end{aligned}$$

Substituting in 25 and 8,

$$\begin{aligned}
0 &= I_1(N - mg) + m\alpha R\left((\mathbf{a} \times (N\hat{\mathbf{z}} + \mathbf{F}_f)) \times \hat{\mathbf{3}}\right) \cdot \hat{\mathbf{z}} + m\alpha R\left(\mathbf{L} \times (\boldsymbol{\omega} \times \hat{\mathbf{3}})\right) \cdot \hat{\mathbf{z}}\\
&= I_1(N - mg) + m\alpha RN\left(-\mathbf{a}(\hat{\mathbf{z}} \cdot \hat{\mathbf{3}}) + \hat{\mathbf{z}}(\mathbf{a} \cdot \hat{\mathbf{3}})\right) \cdot \hat{\mathbf{z}} + m\alpha R\left(-\mathbf{a}(\mathbf{F}_f \cdot \hat{\mathbf{3}}) + \mathbf{F}_f(\mathbf{a} \cdot \hat{\mathbf{3}})\right) \cdot \hat{\mathbf{z}}\\
&+ m\alpha R\left(\boldsymbol{\omega}(\mathbf{L} \cdot \hat{\mathbf{3}}) - \hat{\mathbf{3}}(\mathbf{L} \cdot \boldsymbol{\omega})\right) \cdot \hat{\mathbf{z}} \quad \text{using 12} \qquad (27)\\
&= I_1(N - mg) + m\alpha RN\left(-\cos\theta\mathbf{a} + R(\alpha - \cos\theta)\hat{\mathbf{z}}\right) \cdot \hat{\mathbf{z}} + m\alpha R\left(-F_{f,x}\sin\theta\mathbf{a}\right) \cdot \hat{\mathbf{z}}\\
&+ m\alpha R\left(I_3(\dot{\psi} + \dot{\phi}\cos\theta)\boldsymbol{\omega} - \left(I_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + I_3(\dot{\psi} + \dot{\phi}\cos\theta)\hat{\mathbf{3}}\right)\right) \cdot \hat{\mathbf{z}} \quad \text{as } \mathbf{F}_f \cdot \hat{\mathbf{z}} = 0\text{, and using 20, 22}\\
&= I_1(N - mg) + m\alpha RN\left(-R(\alpha\cos\theta - 1)\cos\theta + R(\alpha - \cos\theta)\right) + m\alpha R\left(-R(\alpha\cos\theta - 1)F_{f,x}\sin\theta\right) \quad \text{using 6}\\
&+ m\alpha R\left(I_3(\dot{\psi} + \dot{\phi}\cos\theta)(\dot{\psi}\cos\theta + \dot{\phi}\right) - \cos\theta\left(I_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + I_3(\dot{\psi} + \dot{\phi}\cos\theta)\right) \quad \text{using 5, 19,3} \qquad (28)
\end{aligned}$$

Gathering terms with and without the normal force in 28, and noting $F_{f,x} = -\frac{v_x\mu_k}{|\mathbf{v_A}|}N$,

$$\begin{aligned}
&I_1 mg - m\alpha RI_3(\dot{\psi} + \dot{\phi}\cos\theta)(\dot{\psi}\cos\theta + \dot{\phi})) + m\alpha RI_3\cos\theta\left(I_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + I_3(\dot{\psi} + \dot{\phi}\cos\theta)\right)\\
=&N\left[I_1 + m\alpha^2R^2(1 - \cos^2\theta) + m\alpha R^2\frac{v_x\mu_k}{|\mathbf{v_A}|}(1 - \alpha\cos\theta)\right]
\end{aligned}$$

Rearranging:

$$\begin{aligned}
N &= \frac{I_1 mg - m\alpha RI_3(\dot{\psi} + \dot{\phi}\cos\theta)(\dot{\psi}\cos\theta + \dot{\phi})) + \cos\theta(I_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + I_3(\dot{\psi} + \dot{\phi}\cos\theta))}{I_1 + m\alpha^2R^2(1 - \cos^2\theta) + m\alpha R^2\frac{v_x\mu_k}{|\mathbf{v_A}|}(1 - \alpha\cos\theta)}\\
&= m\frac{I_1 g + \alpha R\left(I_1\cos\theta(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) - I_3\sin^2\theta\dot{\phi}(\dot{\psi} + \dot{\phi}\cos\theta)\right)}{I_1 + m\alpha R^2\sin\theta\left(\alpha\sin\theta - \frac{\mu_k v_x}{|\mathbf{v_A}|}(1 - \alpha\cos\theta)\right)}
\end{aligned}$$

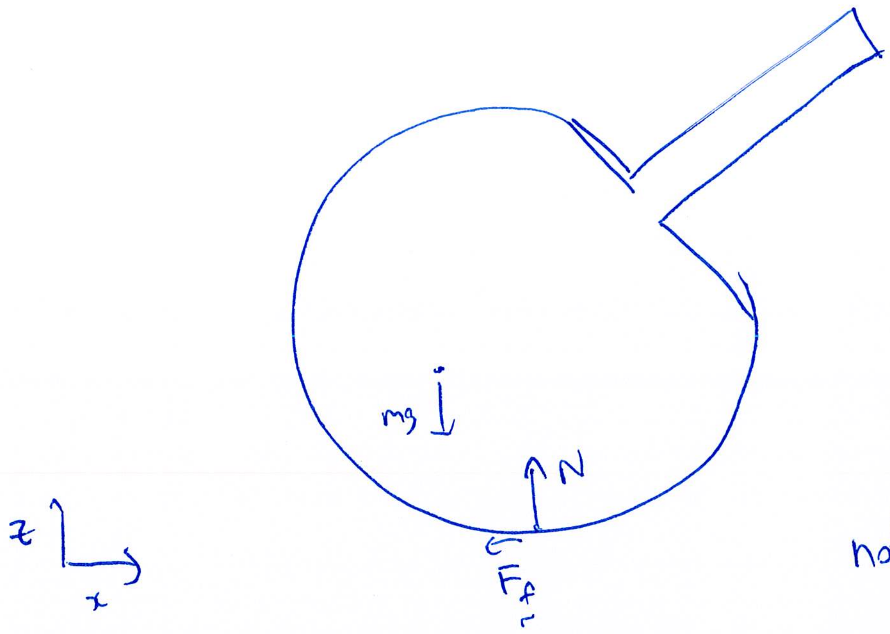
xi. (**1.5 marks**)

The angular momentum about the centre of mass is nonconstant precisely because there is an external torque about the centre of mass. This external torque is given by 14. Thus is must be always orthogonal to the moment arm **a**, the vector from the centre of mass to the point of contact A where the frictional and normal forces apply. Thus, the angular momentum in the direction of **a** is necessarily constant, and $\mathbf{v} = \mathbf{a}$.

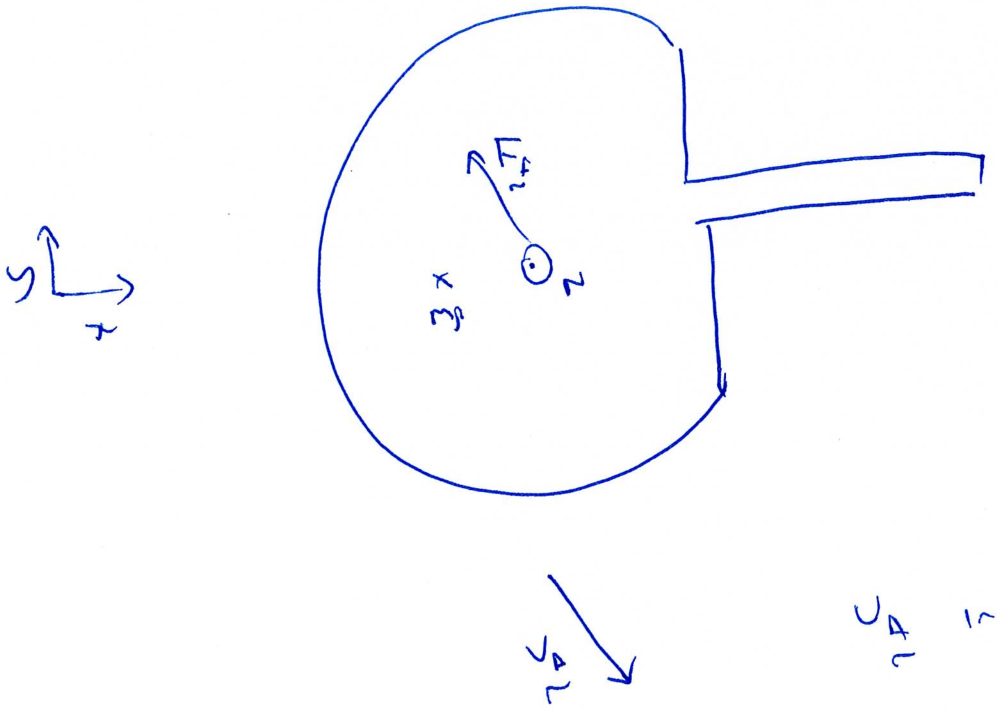
To demonstrate this mathematically, 8, 15, 23 allow

$$\begin{aligned}
-\dot{\lambda} &= \frac{d\mathbf{L}}{dt} \cdot \mathbf{a} + \alpha R\mathbf{L} \cdot \frac{d\hat{3}}{dt}\\
&= (\mathbf{a} \times (N\hat{\mathbf{z}} + \mathbf{F_f})) \cdot \mathbf{a} + \frac{\alpha R}{I_1}\mathbf{L} \cdot (\boldsymbol{\omega} \times \mathbf{L})\\
&= 0
\end{aligned}$$

Q3 A1 SOLN

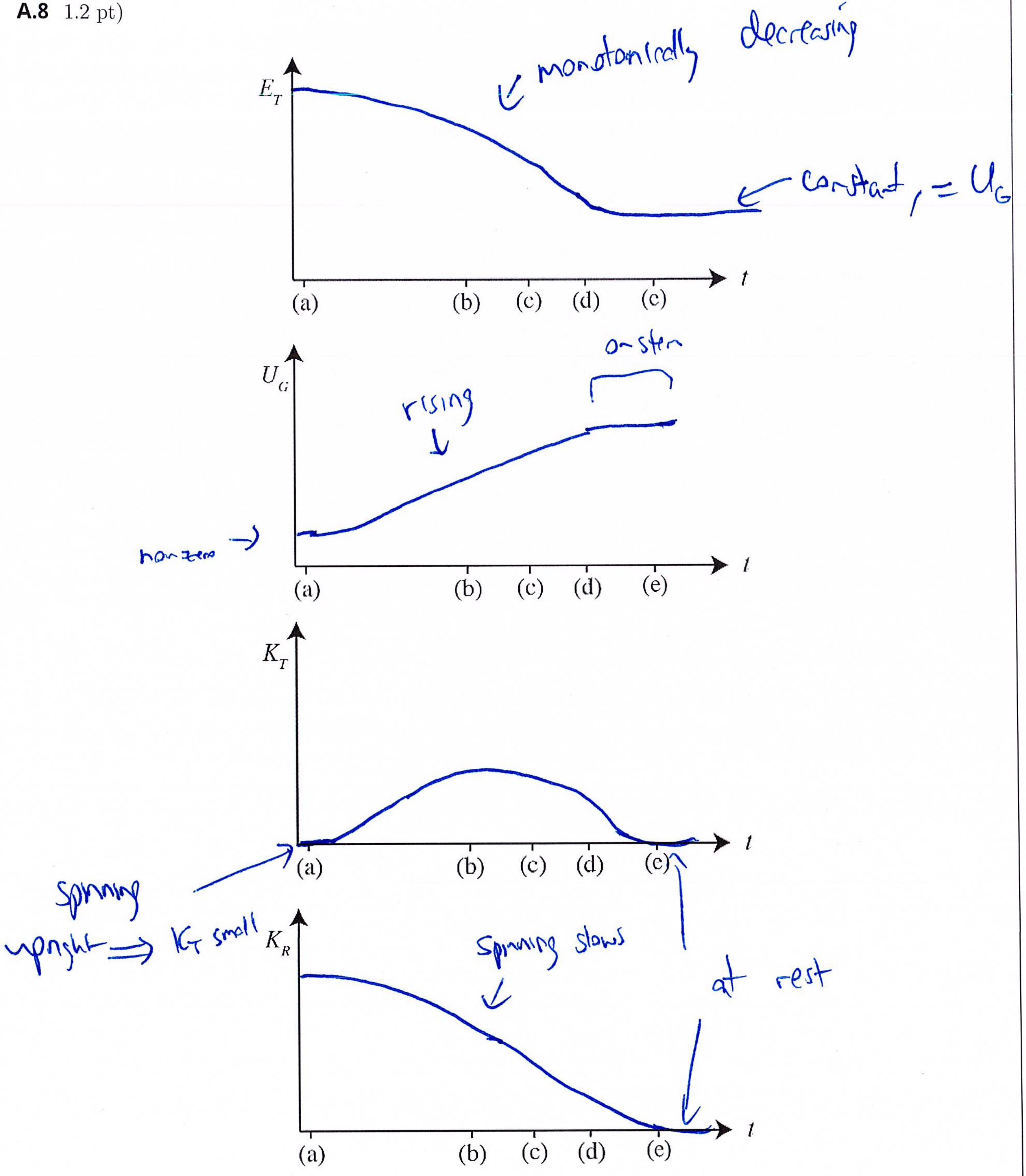


note: direction of F_f unimportant



v_A in direction of $-F_f$

A.8 1.2 pt)


monotonically decreasing
constant, = U_G
E_T
t
(a) (b) (c) (d) (c)
on step
rising
U_G
non zero
t
(a) (b) (c) (d) (e)
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t
(a) (b) (c) (d) (c)
spinning upright ⟹ K_T small
K_R
spinning slows
at rest
t
(a) (b) (c) (d) (e)